On Corners of Objects Built from Parallelepiped Bricks

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Abstract

We investigate a question initiated in the work of Sibley and Wagon, who proved that 3 colors suffice to color any collection of 2D parallelograms glued edge-to-edge. Their proof relied on the existence of an "elbow" parallelogram. We explore the existence of analogous "corner" parallelepipeds in 3D objects, which would lead to 4-coloring. Our results are threefold. First, we refine the 2D proof to render information on the number and location of the 2D elbows. Second, we extend the 2D results to 3D for objects satisfying two properties. Third, we exhibit a genus-0 object (a topological ball) that satisfies one but not both of our properties, and fails the 3D extension theorem, establishing that this theorem is, in a sense, tight.

1 Introduction

Sibley and Wagon [SW00] proved that any collection of parallelograms glued whole-edge to whole-edge must have at least one *elbow*: a parallelogram with two edges incident to one of its vertices *exposed* in the sense that neither is glued to another parallelogram. Elbows have at most two neighbors, which enabled them to prove that such tilings are 3-colorable. The analogous question in 3D is [Wag02, DO03b]: Must every object built from parallelepipeds (henceforth, *bricks*) have at least one *corner*, a brick with three faces incident to one of its vertices, exposed? The bricks must be *properly joined*: each pair is either disjoint, or intersects either in a single point, a single whole edge of each, or a single whole face of each.

Just as in 2D, the existence of corner bricks would lead to 4-colorability of the "brick graph."

But Robertson, Schweitzer, and Wagon [RSW02] found a polyhedron with no corner. Their example, a "buttressed octahedron," has genus 13. Since then, there have been two developments concerning corners of 3D brick objects. First, it was shown in [GO03] that two classes of such objects always have corners: the zonohedra, and objects built from orthogonal bricks, i.e., rectangular boxes. Zonohedra are a particular class of genus-0 objects. Second, we found a genus-3 object with no corners [DO03a], and conjectured that every genus-0 object built from bricks has a corner.

In this paper we refine the 2D results and show that they fail to extend to topological balls in 3D. We deviate from a concentration on genus and instead posit two properties (called S and I/E) such that, if an object possesses these properties, then it must have corners. Finally, we prove that topological balls have the first property (S), but not the second (I/E).

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2 An Object with No Corners: The ZZ-Object

The genus-3 example from [DO03a] serves as a counterexample to many hypotheses, and will be important to illustrate our definitions. The overall design is shown in Fig. 1a. It consists of two Z-shaped paths connecting four cubes. Each of the long connectors has no corner when split lengthwise into four bricks. Similarly, the four cubes have no corners when split into eight cubes. However, as is evident from (a), it is self-intersecting. The self-intersection can be removed

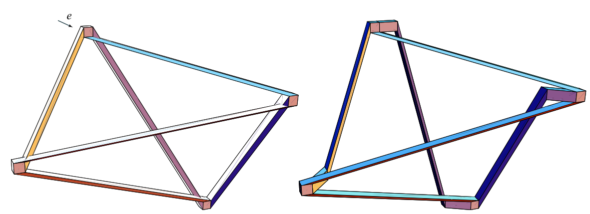


Figure 1: (a) A self-intersecting object with no corners (after refinement) (b) A genus 3 non-self-intersecting object built from 14 bricks with no corners (after refinement).

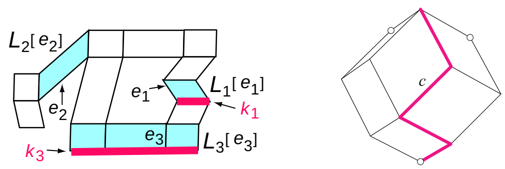
by zig-zagging one of the Zs. The resulting object is shown in Fig. 1b.

3 2D Brick Objects Revisited

Sibley and Wagon [SW00] proved that every 2D object built from parallelograms has an elbow. In this section we refine their theorem both quantitatively and qualitatively. First, we need some definitions.

A *pseudoline* $L[e]$ is a longest sequence of adjacent bricks, all of whose shared edges are parallel to e; it has two "sides" of edges, the *top* and *bottom*. A collapse of a pseudoline $L[e]$ is a chain of edges obtained by shrinking all the edges of $L[e]$ parallel to e to length zero. A *monotone chord* is a boundary-to-boundary chain of edges that is strictly monotone with respect to some direction. Intuitively, a monotone chord z is the result of collapsing a pseudoline $L[e]$.

A path of bricks *crosses* a pseudoline L if it contains a brick of L. A pseudoline L is *separating* (S) if there is no path of bricks connecting a point on its top to a point on its bottom, *without crossing* L; L is called *interior* (I) if none of its side edges are exposed; *exterior* (E) if all of its edges to one side are exposed; and I/E if either I or E. Finally, L is an SI/E-pseudoline, if it is both S and I/E. Similar definitions hold for monotone chords. Fig. 2a shows examples of monotone chords and pseudolines that are interior, exterior, and separating.



(a) (b)

Figure 2: (a) $L_1[e_1]$ and its collapse k_1 are interior and non-separating; $L_2[e_2]$ is interior and separating; $L_3[e_3]$ and its collapse k_3 are exterior (b) A 2D object with three elbows.

An object O is SI/E-*collapsible* if it is either empty, or contains one pseudoline L that satisfies property SI/E, and a collapse of L reduces O to an SI/E-*collapsible* object.

3.1 Refinement of the Sibley-Wagon Theorem

We will distinguish between an *elbow* parallelogram, and an *elbow vertex*, the vertex of the elbow with incident exposed edges. The first extension of Sibley-Wagon is quantitative:

Lemma 1 *Every 2D object built from parallelograms has at least three elbow vertices.*

Proof. Any object O built from parallelograms has an outer boundary P that is a simple polygon. The edges of P are all edges of parallelograms that are on the exterior face (the unbounded component of $\mathbb{R}^2 \setminus O$). Note that P is a simple polygon regardless of the genus of O.

Orient the boundary of P counterclockwise, and define the *turn angle* τ_i at a vertex v_i of P to be the angle needed to turn $v_i - v_{i-1}$ to $v_{i+1} - v_i$; τ_i is π minus the internal angle at v_i.

Let the parallelogram contributing edge $e_i = (v_i, v_{i+1})$ to P have angles α_i at v_i and so $\pi - \alpha_i$ at v_{i+1}. If O has no corners, then every vertex of P must have at least two incident parallelograms of O; for just one incident parallelogram would be a corner. So $\tau_i = \pi - [\alpha_i + (\pi - \alpha_{i-1}) + \beta_i] = \alpha_{i-1} - \alpha_i - \beta_i$, where $\beta_i \geq 0$ is the angular contribution of other parallelograms incident to v_i. We know that for any simple polygon, $\sum_i \tau_i = 2\pi$. In forming this sum via the expression in terms of the α_i's, it is clear that the parallelogram sharing e_i has a net zero contribution, because its terms in τ_i and in τ_{i+1} cancel out. Thus each parallelogram sharing an edge of P contributes 0 to the turn angle sum. We thus have $\sum_i \tau_i = -\sum_i \beta_i \leq 0$. This contradicts the fact that the sum must equal 2π, contradicting our assumption that there are no elbows.

We have now reproved the Sibley-Wagon theorem. But we have more, for each elbow vertex can contribute only strictly less than π to the turn angle (e.g., a sharp parallelogram vertex with nearly 0 internal angle). So, for $\sum_i \tau_i$ to reach 2π, we need at least three elbow vertices. If these three elbow vertices are vertices of distinct elbows, we are finished. Only a one-parallelogram object can have three elbow vertices on one elbow, and here the lemma is clearly true. So assume there is an elbow with two elbow vertices on P. Then together those two vertices contribute exactly π to the turn, still leaving a gap to reach 2π. So the claim of the lemma is established. □

That this result is best possible is established by Fig. 2b. The second extension of Sibley-Wagon is qualitative, yielding information about where the elbows are: one to each side of every monotone chord. We can prove this using turn angles, without other assumptions. We choose instead to prove a weaker version which extends almost directly to 3D.

Lemma 2 *A collapse of a pseudoline is a monotone chord.*

Lemma 3 *A monotone chord can cross a pseudoline at most once.*

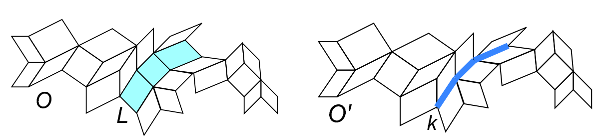
Lemma 4 *If O' is an object obtained by collapsing a pseudoline in O, then a monotone chord k in O reduces to a monotone chord k' in O'.*

Theorem 5 *For any SI/E-collapsible object O and any SI/E-monotone chord z of O, there is an elbow vertex strictly to each nonempty side of z.*

Proof. The proof is by induction on the number of pseudolines. The base case is an object with one pseudoline L. By Lem. 3, a monotone chord z may cross L at most once. This implies that a nonempty side of z will contain at least one of the two end bricks of L, each of which has two elbow vertices, at most one of which may be on z.

The inductive hypothesis is that the theorem holds for any SI/E-object with N or fewer pseudolines. To prove the inductive step, we consider an arbitrary SI/E-object O with $N+1$ pseudolines, and reestablish the inductive hypothesis for O.

Pick a SI/E-pseudoline L whose collapse reduces O to SI/E-collapsible object O'. Note that O' may self-intersect geometrically, but we are only concerned with the combinatorial structure and the parallelism of edges; global intersections are irrelevant for our property. (Alternatively, one can view bricks on either side of L to be topologically extended through L.) By Lem. 2, L reduces to a monotone chord k in O', as illustrated in Fig. 3.

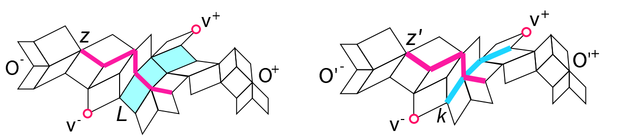


(a) (b)

Figure 3: (a) L is a SI/E-pseudoline (b) The collapse k of L is a monotone chord.

Let z be an arbitrary SI/E-monotone chord in O and let O^+ and O^- be the pieces of O separated by z. By Lem. 4, the monotone chord z reduces to a monotone chord z' in O'. Let O'^{+} and O'^{-} be the pieces to which O^{+} and O^{-} reduce in O'.

Next we focus on O^{+} and show that if nonempty, it contains an elbow vertex v^{+} not on z. The inductive hypothesis applied to O' tells us that O'^{+} has an elbow vertex v not on z. If this v is away from the collapse, it can serve to establish the claim; but if it is involved in the collapse, further argument is needed. We now consider two cases, depending on whether L is interior or exterior. We only detail the first case in this abstract.



(a) (b)

Figure 4: (a) L is interior; z is a SI/E-monotone chord (b) Elbows in O' are elbows in O.

If L is interior, then k is interior, with none of its vertices exposed. Refer to Fig. 4. This implies that v is not on k; consequently, v is not on L. Then $v^{+} \equiv v$ is exposed in O^{+}; this along with the fact that v does not touch z settles this case. □

4 Extensions of Results to 3D

The result of Thm. 5 extends to 3D SI/E-*collapsible* objects. First we need some definitions.

A *pseudoplane* $Z[e]$ is a collection of bricks, all with edges parallel to e, defined recursively as follows: (1) Every brick that shares e is in $Z[e]$ (2) If b is in $Z[e]$ and e' an edge of b parallel to e, then every brick that shares e' is in $Z[e]$. Note that the cornerless ZZ object (Fig. 1) is in fact a single pseudoplane $Z[e]$. Although a pseudoplane is connected, its interior may not be connected, i.e., there could be "pinchings" at edge-to-edge contacts between bricks. The *top* and *bottom* of $Z[e]$ are defined in the obvious way: the collection of all faces of bricks of $Z[e]$ that have no edge parallel to e, and are incident to the top or bottom endpoint of e, respectively.

A *collapse* of a pseudoplane $Z[e]$ is a sheet of faces z obtained by shrinking all edges in $Z[e]$ parallel to e to zero length. A *monotone sheet* z is, intuitively, the result of collapsing a pseudoplane $Z[e]$. The top and the bottom of a pseudoplane are both monotone sheets.

We define a pseudoplane Z to be *separating* (S), *interior* (I), *exterior* (E), I/E, SI/E, exactly analogously to the 2D definitions for pseudolines. Similarly, a 3D object O is SI/E-*collapsible* just as in 2D.

Thm. 6 is a direct extension of Thm. 5, and its proof follows the 2D proof closely:

Theorem 6 *For any monotone sheet s of a SI/E-collapsible object, there is a corner vertex strictly to each nonempty side of s.*

Corollary 7 *Any SI/E-collapsible object has at least three corners.*

5 Topological Balls

In this section we focus on topological balls built from bricks and show that they satisfy property S, but not property I/E.

Lemma 8 *Any pseudoplane of a topological ball O is separating and O is S-collapsible.*

Lemma 9 *Topological balls are not I/E-collapsible.*
This last claim is established by the example in Fig. 5, which shows a topological ball O composed of a pseudoplane $Z = Z[e]$, with e a vertical edge, and a symmetric cornerless piece O^{+} that lies on top of Z ($O = Z \cup O^{+}$). Fig. 6 shows top and bottom views of O. O is composed of

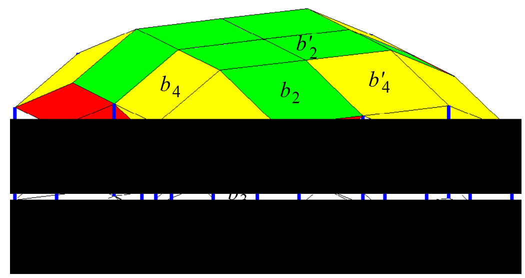


Figure 5: Side view of a topological ball O with no corners on one side of the pseudoplane $Z = Z[e]$.

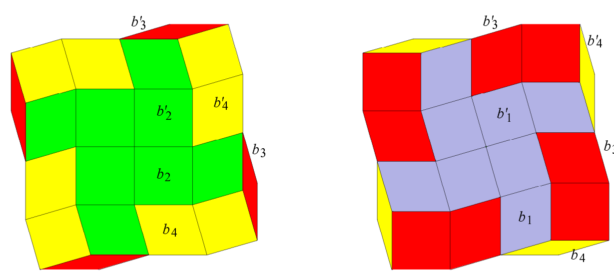


Figure 6: Top (left) and bottom (right) of the object O from Fig. 5.

four identical smaller pieces glued together, with one piece Q as illustrated in Fig. 7b. Each piece Q is composed of four mutually adjacent bricks, with Z adjacent to three of them (see Fig. 7a).

This object O shows that the 2D result of Thm. 5 does not extend to 3D. Indeed we can show that the 3D extension, Thm. 6, is best possible in the following sense. O satisfies property S by Lem. 8, but it is not I/E-collapsible, in that, after collapse of Z, the remaining object O^{+} does not satisfy property I/E. Fig. 8a delineates an arbitrary pseudoplane $A(a)$ of O, whose collapse induces a new corner u not present in O. It is just this situation that blocks the proof technique employed in Thms. 5 and 6.

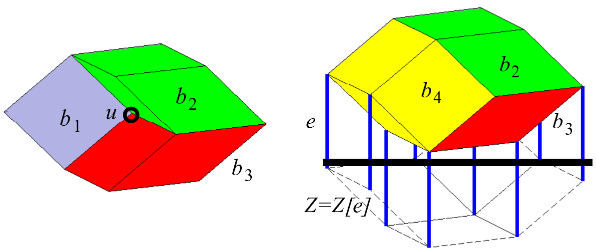


Figure 7: (a) Three mutually adjacent bricks share convex vertex u (b) Four mutually adjacent bricks share u and sit on top of the pseudoplane $Z[e]$.

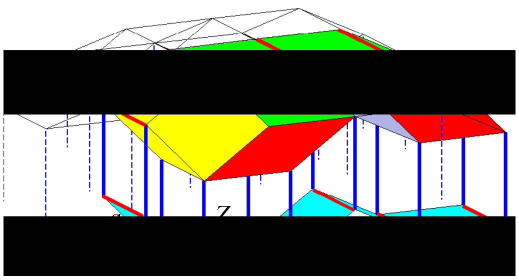


Figure 8: A pseudoplane $A(a)$ of O whose collapse induces a new corner u.

6 Conclusions

We have shown that there exist topological balls with no corners to one side of a pseudoplane, unlike the analogous situation in 2D. We identified a class of 3D objects, which we called `SI/E`-collapsible, that do have corners on each side of any pseudoplane, and showed the result tight in some sense. This proves the existence of corners, and therefore 4-colorability, for `I/E`-collapsible topological balls. (All objects built from bricks are known to be 5-colorable [RSW02] [GO03]).

Although it is tempting to glue several copies of the object from Fig. 5 together to build a cornerless ball, we have not been able to complete this construction. Thus the main question raised in [RSW02]—whether every ball has at least one corner—remains open.

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