# Unfolding Polyhedral Bands

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**Abstract**

A *band* is defined as the intersection of the surface of a convex polyhedron with the space between two parallel planes, as long as this space does not contain any vertices of the polyhedron. An *unfolding* of a given band is obtained by cutting along exactly one edge and placing all faces of the band into the plane, without causing intersections. We prove that for a specific type of band there exists an appropriate edge to cut so that the band may be unfolded.

## 1 Introduction

It has long been an unsolved problem to decide whether every polyhedron may be cut along edges and unfolded flat to a single, nonoverlapping polygon [7, 5, 4]. An interesting special case emerged in the late 1990s: [1] can the *band* of surface of a convex polyhedron enclosed between parallel planes, and containing no polyhedron vertices, be unfolded without overlap by cutting a single edge? A band and its associated polyhedron are illustrated in Fig. 1.

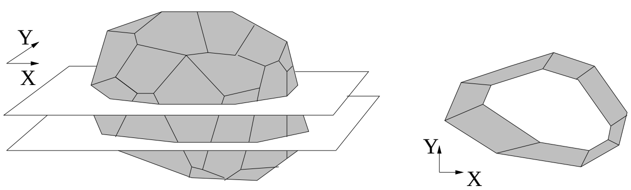


Figure 1: A polyhedron cut by two parallel planes, and a top view of the resulting band.

This band forms the side faces of what is known as a *prismadoid*—the convex hull of two parallel convex polygons in $\mathbb{R}^3$—but the band unfolding question ignores the top and bottom faces $A$ and $B$ of the prismatoid. An example was found (by E. Demaine and A. Lubiw) that shows that band unfoldings can overlap, if a "bad" edge is chosen to cut; see Fig. 2.

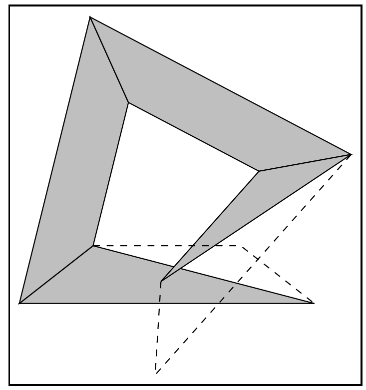

Figure 2: A truncated tetrahedron can unfold to an overlapping configuration if the wrong edge is cut.

So the question remained: Does there always exist a "good" edge to cut? This paper answers YES for a special case: when the top $A$ and bottom $B$ polygons of the band are *nested* in the sense that the projection of $A$ onto the plane of $B$ falls strictly interior to $B$. In this case we say that the band is *nested* (as shown in Fig. 1). Intuitively, we might expect to obtain a nested band if both planes cut the polyhedron near its "top". Our argument provides more than nonoverlap in the final planar state: it ensures non-intersection throughout a continuous unfolding motion. Moreover, we believe the argument should extend to capture arbitrary bands.

Band-like constructs have been studied before. Bhattacharya and Rosenfeld [2] define a polygonal *ribbon* as a finite sequence of polygons, not necessarily coplanar, such that each pair of successive polygons intersects exactly in a common side. Triangular and rectangular ribbons (both open and closed) have also been studied. Arteca and Mezey [1] deal with continuous ribbons. Simple bands can be used as linkages to transfer mechanical motion, as pointed out by Cundy and Rollett [3].

There is one unfolding result that is relevant to our problem, which may be interpreted as unfolding infinitely thin bands. This is that a "slice curve," the intersection of a plane with a convex polyhedron, develops in the plane without overlap [6]. This holds regardless of where this curve is cut. Thus, both the top and the bottom boundary of any band (and in fact any slice curve between), unfold without overlap. So overlap can only occur from interaction with the cut

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[1] Posed by Erik Demaine, Martin Demaine, Anna Lubiw, and Joseph O'Rourke, 1998.

edge, as in Fig. 2.

## 2 Unfolding Nested Bands

The projection of a nested band has certain properties. Every vertex is incident to exactly three edges. Two of these edges belong to one of the nested polygons, and the third links to the other polygon. There are no edge crossings. The projection is partitioned into quadrilaterals, each corresponding to part of a polyhedral face. Since each face is flat, the quadrilaterals in are in fact trapezoids where edges from the inner and outer polygons are parallel. Unless mentioned otherwise, all arguments in this section involve the projection of a band.

We continue with some definitions that are necessary to describe the unfolding motions:

An edge of a band is a *hinge* if it was part of an edge of the given polyhedron. All hinges have an endpoint on each of the given parallel planes.

After *cutting* a single hinge, a *flattening motion* is a continuous motion during which each face moves rigidly but remains connected to each adjacent face via their common hinge, and the resulting configuration is planar. If no intersection occurs during the motion, then this motion is an *unfolding*.

A planar chain is *convex* if joining the endpoints with an edge yields a convex polygon.

A non-convex chain is *weakly convex* if we encounter only left (or only right) turns as we traverse it, *and* joining the endpoints with an edge yields a polygon which has no exterior angles less than $\frac{\pi}{2}$.

Any chain that has only left (right) turns but is not convex or weakly convex is a *spiral*.

The *interior angle* at a vertex of a spiral or convex chain is the smaller of the two angles at the vertex. Exterior angles are defined accordingly.

The *normal cone* of a vertex $v$ belonging to a convex polygon is the region between two halflines that begin at $v$, are respectively perpendicular to the two edges incident at $v$ and are both in the exterior of the polygon.

We say that a point is to the left (right) of a segment $xy$ if it is to the left (right) of a directed line through $xy$.

When an edge of a given band is cut, the two convex polygons in the projection mentioned above are cut into (degenerate) convex chains. Suppose that we begin a flattening motion by "squeezing" the two parallel planes and keeping all vertices of the band on the planes. Such a motion will increase the interior angle at every vertex in the projection. Furthermore an interior angle can only open to $\pi$. Thus in the projection a convex chain cannot self-intersect after such a motion. Proofs for these claims are omitted here. All of our proofs involve this specific method of flattening.

The projection of a band is related to the actual flattening motion of a band as follows: let $z_0$ be the vertical separation between the two planes, $A$ and $B$. The two planes will move towards each other, always remaining parallel. Vertices of the hinges will always remain in the planes. Let $f$ be one band face, the hull of parallel edges $b_1b_2$ and $a_1a_2$. $z_0$ is determined by the dihedral angle at $b_1b_2$ between $f$ and the base plane $B$. At any one time, the $2D$ picture is an overhead projection of the $3D$ band, with $z$ decreasing from its initial value $z_0$ to 0, at which time it is entirely flat in the $B$ plane, i.e. we have a uniform squashing of the band by lowering $A$ until it meets $B$. For any face $f$, the value of $z$ determines its dihedral angle with respect to $B$. The opening of the convex chains, visible in the overhead view represents the turning at each hinge, necessary to accommodate the various simultaneous dihedral motions.

Let the vertices of the inner polygon be ordered in clockwise order, and the cut hinge be incident to vertex $a_i$. We hold $a_{i-1}a_i$ fixed horizontally in the plane and relabel the newly created endpoint as $a^*$. Correspondingly, for the outer polygon, the direction of $b_{i-1}b_i$ remains fixed (it moves away from $a_{i-1}a_i$ but remains parallel) and $b^*$ is a "moving" endpoint. Thus the cut edge is split into edges $a_ib_i$ and $a^*b^*$. These definitions are illustrated in Fig. 3.

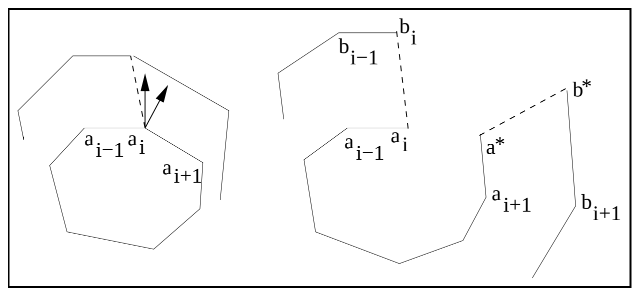


Figure 3: Left: projection of the inner convex chain and part of the outer chain. The cone of a vertex $a_i$ is shown, as well as the projection of the polyhedral edge incident to $a_i$. Right: the result of cutting at $a_ib_i$ and flattening.

Notice that the projection of the hinge incident to $a_i$ becomes longer after flattening.

**Lemma 1** *A flattened band cannot produce an inner chain that is a spiral.*

**Proof.** During our type of flattening motion $a^*a_{i+1}$ can only rotate clockwise, because all joints open clockwise, and the centers of rotation at these joints are all left of $a_{i+1}a^*$. Let $R$ be the region that is to the right of the two half-lines that form the normal cone of $a_i$. As the unfolding motion begins, $a^*$ can only move within $R$. This can be seen by opening each angle successively in clockwise order, starting with the angle at $a_{i+1}$. Also, it follows from Cauchy's arm lemma (see, e.g., [8]) that no two points on an opening convex chain approach each other. Eventually $a^*$ may end up anywhere within $R$ *or* in the region to the right of $a_{i-1}a_i$, but only after tracing a clockwise motion about $a_i$. Consequently, a flattened band cannot produce a spiral. Also notice that the final direction of $a^*b^*$ will always be more clockwise than the final direction of $a_ib_i$. □

**Lemma 2** *If a flattening produces an inner chain that is convex then the band can be unfolded.*

**Proof.** If intersection is to occur, then some part of the inner chain must cross through $a_ib_i$ *or* $a^*b^*$. This follows from the results on slice curves, mentioned in the introduction. From the arguments of the previous lemma, we see that the inner chain will be convex throughout the motion. Since the direction of $a^*b^*$ is always more clockwise than that of $a_ib_i$, the ends of the band cannot intersect. □

The same types of arguments may be used to prove that we can safely cut along any hinge where $b_i$ is located within the normal cone of $a_i$, or any hinge incident to an acute interior angle.

We now characterize the types of chains that may be obtained after a flattening resulting from a cut at $a_ib_i$. We say that a chain is "safe" if it is convex. There are two types of "dangerous" chains, depending on which endpoint is not on the hull (clearly one of the two endpoints must be on the hull). Suppose that $a_i$ is not on the hull of the opened chain. A problem might arise if $a_ib_i$ was initially to the right of the normal cone at $a_i$. In other words, $a_*$ might cross through $a_ib_i$. In this case the chain is "unsafe" (see Fig. 4). We note that if the whole flattening motion is observed, it is possible that this crossing might happen but in the final position there will be no intersection (i.e. $a_*$ and all successive edges might cross out again). In other words, the term "unsafe" serves just as a warning. Even under these conditions there may be no intersection at any time of the flattening motion.

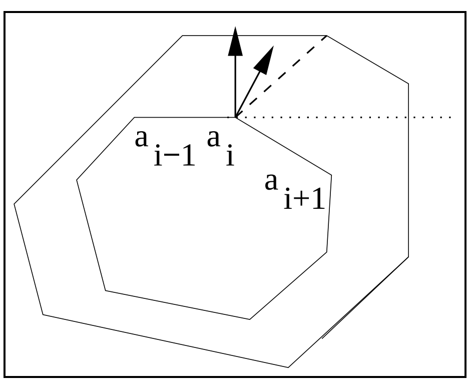


Figure 4: Cutting at $a_i$ is "dangerous" if $a^*$ ends up above the dotted line. In this case the cut is labeled "unsafe" if the hinge at $a_i$ (shown dashed) is to the right of its normal cone. A symmetric dangerous and unsafe case exists for the other side of the cone.

As mentioned, there are two types of dangerous unfoldings, and in each case there is only a potential problem if a hinge lies on a specific side of its associated normal cone. Clearly if a given band cannot unfold with our motion then all vertices are associated with unsafe openings.

**Lemma 3** *Not all hinges can be to the left (or all to the right) of their associated normal cones. Thus not all vertices can have the same type of unsafe property.*

**Proof.** It is enough to look at the initial projection to see this: suppose without loss of generality that on the inner chain all hinges are clockwise of their respective normal cones. Take any trapezoid with height $h$ (measured in the projection). The trapezoid belonging to the next edge clockwise must have height greater than $h$. This continues around the convex polygon until we reach the original trapezoid which would have to have height greater than $h$. So somewhere there is a vertex $a_k$ whose hinge is counterclockwise of the normal cone at $a_k$, while the hinge at $a_{k+1}$ is clockwise of its respective cone. □

Suppose that we have located two successive vertices as described in the previous lemma. For the cuts at both vertices to be unsafe, in each case some portion of edge $a_ka_{k+1}$ is not on the resulting hull of the inner chain (see Fig. 5). In other words the type of *dangerous* opening cannot be the same at both vertices.

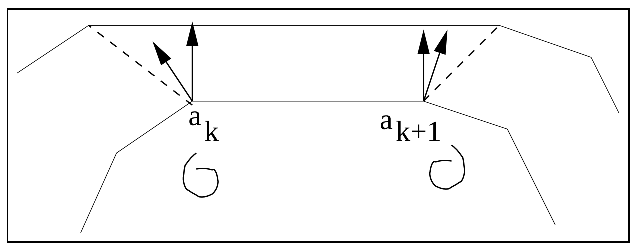


Figure 5: The type of dangerous opening (indicated by the curves below the labeled vertices) must alternate between some pair of successive vertices.

**Lemma 4** *Cutting a hinge incident to either $a_k$ or $a_{k+1}$ (defined in the previous lemma) must result in a chain that is not unsafe.*

**Proof.** Let us begin by cutting at $a_{k+1}$. As usual, we hold $a_ka_{k+1}$ fixed horizontally and open all angles. Newly created $a^*$ must end up in the upper-right quadrant of $a_{k+1}$, in order to have the necessary type of dangerous opening. Now we make a new cut at $a_k$, and translate the entire unfolded chain (except the fixed edge) so that $a^*$ re-attaches to $a_{k+1}$. We let the translated copy of $a_k$ retain its label, and call the horizontal edge $a^*a_{k+1}$. Notice that $a_k$ must be in the lower left quadrant of $a^*$ (see Fig. 6).

Now we have a new opened chain, except that we have not taken care of the openings at the angles of $a_k$ and $a_{k+1}$. Since $a_{k+1}a_{k+2}$ (previously $a^*a_{k+2}$) had rotated clockwise in the first unfolding, and we have merely translated it back,

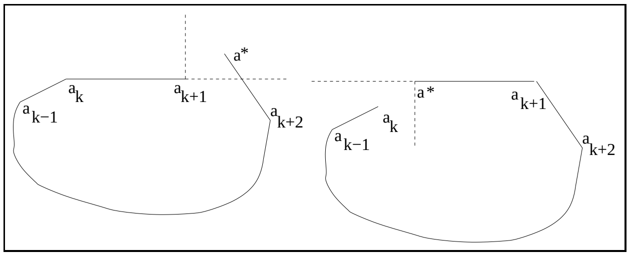


Figure 6: Left: an unfolded chain. Right: translating part of the chain so that the cut vertex is switched.

we must rotate it counterclockwise to return it to its initial orientation. We must then further rotate it counterclockwise in order to open the interior angle at $a_{k+1}$. The entire chain will rotate rigidly as well. Thus $a_k$ cannot cross into the upper-left quadrant of $a^*$. Now notice that during the first opening, edge $a_{k-1}a_k$ rotated clockwise, due to the opening of the angle at $a_k$. So we might expect that in order to compensate for this in our final diagram we should rotate $a_{k-1}a_k$ counterclockwise (which might cause $a_k$ to go above the horizontal line). After all, if a cut is made at $a_k$, then $a_{k-1}a_k$ must rotate counterclockwise from its initial position, but now it is clockwise. However, since the opening of the angle at $a_{k-1}$ was included in the first opening, and this has not been tampered with, then edge $a_{k-1}a_k$ must be in its correct position. The counterclockwise motion produced by adjusting the angle at $a_{k+1}$ is enough to make the direction of $a_{k-1}a_k$ more counterclockwise than it was initially.

This means that cutting at $a_k$ leads either to the same type of dangerous opening as $a_{k+1}$ or to a safe opening. We conclude that an opening which is not unsafe exists either at $a_k$ or at $a_{k+1}$ □

Since we can always find a vertex to cut so that the inner chain opens to a position that is not unsafe, we can always find an edge to cut along so that a nested band can be unfolded:

**Theorem 5** *Every nested band can be unfolded.*

## 3 Remarks

In a *closed band*, vertices are allowed on the parallel planes of the slab. We claim that all *closed* nested bands may also be unfolded, though proof is omitted here. We also believe that a more complex proof establishes that all bands may be unfolded. Even with it established that arbitrary bands may be unfolded without overlap, it remains interesting to see if this will lead to an unfolding of prismatoids without overlap, including the top and bottom polygons $A$ and $B$. It is natural to hope they could be nestled on opposite sides of the unfolded band, but it is not obvious how to ensure nonoverlap.

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