Banana Spiders: A Study of Connectivity in 3D Combinatorial Rigidity

Andrea Mantler *   Jack Snoeyink †

Abstract

Finding a combinatorial test for rigidity in 3D is an open problem. We prove that vertex connectivity cannot be used to construct such a test by describing a class of mechanisms that increase the vertex connectivity of flexible graphs to 5. Our result is tight, as minimally rigid graphs in 3D can be at most 5-connected.

1 Introduction

In two dimensions, combinatorial rigidity is well understood: Laman's condition on the number and distribution of edges is both necessary and sufficient for determining if a framework is rigid. In three dimensions, however, finding a test for combinatorial rigidity has proved elusive. Little has been published on the failed attempts. In this paper we show that vertex connectivity does not help us in our goal: 3-connectivity together with the 3D extension to Laman's condition is insufficient, and 4- and 5-connectivity are neither sufficient nor necessary; a minimally rigid graph cannot be greater than 5-connected.

There are many models of rigidity. We examine first-order rigidity of bar-joint frameworks [3, 5]. Mathematically, a framework is defined as graph with an embedding in $\Re^d$. Once embedded, the edges of the graph become fixed length bars connected at flexible joints. Knowing whether a framework is flexible or rigid, i.e. whether or not there exists an edge-length preserving deformation that changes the distances between some non-adjacent vertices, is useful in many applications, such as designing bridges and other structures. If a graph G has a rigid embedding, then almost all embeddings of G produces a rigid framework. Thus we would like to assume a *generic embedding* (see [3, 5]), and determine whether or not a framework is rigid based solely on the graph of vertices and edges. (We call a graph *rigid* in $\Re^d$ if there exists an embedding in $\Re^d$ that gives a rigid framework.)

In 1970, Laman published a condition that can be used to test whether a graph is rigid in $\Re^2$:

Condition 1 (Laman, [3, 4]) *A graph $G = (V, E)$ is rigid for dimension 2 if and only if there is a subset E' of E such that:*

1. $|E'| = 2|V| - 3$*, and*
2. *for all* $E'' \subset E'$ *where* $|V(E'')| \geq 2$*, we have* $|E''| \leq 2|V(E'')| - 3$.

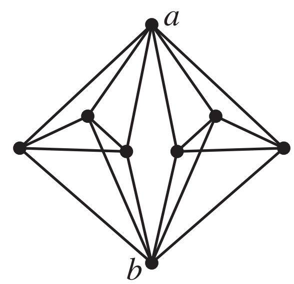

Figure 1: The double banana, with an implied hinge edge through a and b.

This condition, known as Laman's condition, is both necessary and sufficient. Note that the graph $G' = (V, E')$ is *minimally rigid*: removing any edge from G' gives a flexible graph. Embedded generically, a minimally rigid graph produces an isostatic framework [5].

Modifying Laman's condition for 3D, we get:

Condition 2 ([3]) *A graph $G = (V, E)$ is rigid for dimension 3 if and only if there is a subset E' of E such that:*

1. $|E'| = 3|V| - 6$*, and*
2. *for all* $E'' \subset E'$ *where* $|V(E'')| \geq 3$*, we have* $|E''| \leq 3|V(E'')| - 6$.

We refer to Condition 2 as Laman's condition, and call graphs satisfying this condition *Laman graphs*. Although Laman's condition is necessary, it is no longer sufficient. The *double banana* [2], shown in Figure 1, is the classic example of a framework that satisfies Laman's condition, yet is flexible.

The double banana is the smallest example where Laman's condition is insufficient, but what are others? Lacking a necessary and sufficient extension of Laman's condition to 3D, we would at least like to characterize the cases where Laman's condition is not sufficient.

A natural question is whether triangles are required for rigidity. Euler's formula shows that planar graphs require at least one triangle to be rigid in 2D, and must be fully triangulated to be rigid in 3D. The bipartite graph $K_{3,3}$, however, was known in the 19th century to be infinitesimally rigid in 2D. Bolker and Roth [1] proved that triangles are also not

*Department of Computer Science, University of North Carolina at Chapel Hill, mantler@cs.unc.edu

†Department of Computer Science, University of North Carolina at Chapel Hill, snoeyink@cs.unc.edu

necessary for rigidity for non-planar graphs in 3D: bipartite graphs $K_{4,6}$ and $K_{5,5}$ are generically rigid in 3D, as are all bipartite graphs $K_{m,n}$ where $m, n \geq 5$.

In this paper we extend the double banana counterexample to show that vertex connectivity together with Laman's condition is neither a necessary nor a sufficient condition for rigidity in three dimensions.

In the double banana, there is an implied hinge through vertices a and b. A *hinge* is an edge, $h_{ij} \in E$, around which two or more rigid components can rotate. An *implied hinge*, $h_{ij} \notin E$, consists of a pair of vertices, v_i and v_j, whose distance is implied by two or more disjoint (except in v_i and v_j) maximal rigid components. The line through v_i and v_j acts as the hinge pin around which the attached rigid components rotate.

The double banana is a 2-vertex-connected graph. A graph is *k-vertex-connected* (or *k-connected*) if there exist k vertices such that removing these vertices disconnect the graph, but no set of $k-1$ vertices disconnect the graph. A natural, but false, conjecture is that graphs with implied hinges are at most 2-connected. We add mechanisms (*spiders*) to increase the connectivity of any graph with an implied hinge.

2 Upper Bound on Connectivity

To begin, we observe that minimally rigid graphs cannot have 6-vertex connectivity or higher.

Theorem 1 *A minimal graph, $G = (V, E)$, that satisfies Laman's condition is at most 5-connected.*

Proof. Vertex degrees in a k-connected graph are at least k, and thus $|E| = \frac{1}{2}\sum_{v \in V} deg(v) \geq k|V|/2$. Since $|E| = 3|V| - 6$, we get $k \leq (6|V| - 12)/|V|$, and thus G is at most 5-connected. □

We are interested, therefore, in the possibility of 3-, 4- and 5-connected flexible graphs.

3 A 3-Connected Flexible Graph

Figure 2(a) from Whiteley's survey [5] illustrates the simplest spider that converts the 2-connected double banana to a 3-connected flexible graph. This spider consists of a single vertex, v_0, connected by three edges (*legs*) to the two bananas. Notice that we do not connect the legs to the implied hinge vertices. As the bananas rotate, vertices v_1 and v_2 move closer or farther apart, causing v_0 to swing up or down.

Lemma 2 *The graph $G = (V, E)$ in Figure 2(a) is a 3-connected, flexible Laman graph.*

Proof. The reader can check that the graph G is 3-connected, as it has no cut set of size two, and that it satisfies Laman's condition, as adding one vertex and three edges maintains $|E| \leq 3|V| - 6$ for all induced subgraphs, with equality for the full graph.

Adding the basic spider adds one vertex and three edges, maintaining the equation $|E| = 3|V| - 6$ by adding three to each side. No subgraph violates part 2 of Laman's condition, thus, G continues to satisfy Laman's condition.

To verify the graph is still flexible, we look at the space of infinitesimal motions, which is a linear subspace. Adding an edge adds a single linear constraint, reducing the dimension of the subspace by 1. The space of motions of a graph with a hinge plus the spider body has dimension at least 10: 3 for the Euclidean degrees of freedom for the spider body vertex, 6 for the Euclidean degrees of freedom for the graph, and one for the flexibility at the hinge. Adding the three spider legs reduces the dimension to 7. Thus, there is one internal degree of freedom, and the graph with the spider is flexible. □

We now move on to flexible graphs with higher connectivity.

4 A 4-Connected Flexible Graph

Figure 2(b) shows an example of a 4-connected flexible graph that satisfies Laman's condition. In this graph, we have a spider with a triangular body, and six legs connecting the spider body to the double banana. Notice that we have the spider legs connecting to non-hinge vertices, three legs per banana, with the legs for each banana terminating in two vertices.

Lemma 3 *The graph $G = (V, E)$ in Figure 2(b) is a 4-connected, flexible Laman graph.*

Proof. As in the proof of Lemma 2, observe G with the spider, G_s, removed: $G_b = G - G_s$ is the two-connected double banana shown in Figure 1. The set $\{a, b\} \subset V$ is the only cut set of size two of G_b, and there are no cut sets of size three.

Adding the spider back to G_b, we see that the set $\{a, b\} \subset V$ no longer forms a cut set, and we must remove another two vertices in order to disconnect G. Additionally, we cannot disconnect any component of G_s from G without removing at least four vertices. Graph G is 4-connected.

Adding G_s adds three vertices and nine edges, maintaining the equation $|E| = 3|V| - 6$ by adding nine to each side. No subgraph violates part 2 of Laman's condition, thus, G continues to satisfy Laman's condition.

The space of motions for G_b plus a triangle (the spider body) has dimension 13, since a triangle has 6 Euclidean degrees of freedom. Adding the six legs reduces the dimension to 7, and again the graph with the spider remains flexible. □

5 A 5-Connected Flexible Graph

Figure 2(c) illustrates an example of a 5-connected flexible graph that satisfies Laman's condition. In this graph, the spider body has grown to 6 vertices, and forms a minimally rigid

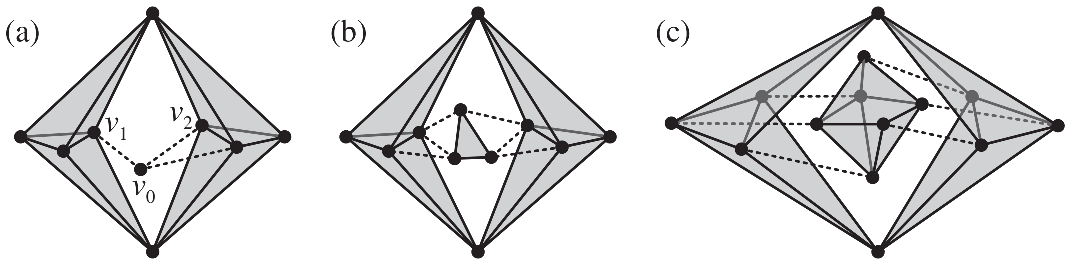

Figure 2: The double banana with spiders increasing the vertex connectivity to (a) 3-connectivity, (b) 4-connectivity, and (c) 5-connectivity. Spider legs are drawn as dashed lines.

graph. Notice that within the body, each vertex has degree 4. We add one leg to each spider body vertex, increasing the degree of each body vertex to 5. Three legs connect to each banana of the double banana, and each leg terminates at a distinct, non-hinge vertex.

Lemma 4 *The graph $G = (V, E)$ in Figure 2(c) is a 5-connected, flexible Laman graph.*

The proof of Lemma 4 is basically the same as that of Lemma 3.

6 Beyond Bananas

In this section we give a method to increase the connectivity of a graph, $G = (V, E)$, without decreasing the flexibility, or causing Laman's condition to be violated. To increase the connectivity, we add copies of the 5-spider described in Section 5.

Let $V = \{v_0, \ldots, v_{n-1}\}$ be the vertices of G. We add n spiders, with spider i having feet $v_i, \ldots, v_{i+5}$, where the indices are taken modulo n. We will now prove that this graph, G_5, consisting of G plus n spiders, is 5-connected. That is, that there are no *bad cut sets*, which are cut sets with fewer than five vertices.

Lemma 5 *Any graph G can be embedded as a vertex-induced subgraph of a 5-vertex-connected graph $G_5 = (V_5, E_5)$.*

Proof. Because every vertex has degree at least five, no bad cut set can isolate a single vertex.

If there is a bad cut set, V_c, containing a spider body vertex, v_s, then there is a bad cut set V_c' that includes vertices only from V. Cut set V_c splits V_5 into V_1 and V_2. We prove that $V_c' = V_c - \{v_s\} + \{v_f\}$ is also a bad cut set, where v_f is the foot vertex adjacent to v_s. This results in a cut set with one fewer spider body vertices. By induction, we can find a V_c' that contains vertices only from V.

Besides v_f, the only neighbours of v_s are four spider body vertices, since we always connect new spiders to vertices of G. The spider body neighbours of v_s cannot be in both V_1 and V_2, since then there would be edges connecting V_1 and V_2. Thus, w.l.o.g., v_f is in V_1, the spider body neighbours of v_s are in V_2 and V_c, and moving v_f to V_c and v_s to V_2 results in a valid bad cut set.

Finally, we prove that a bad cut set $V_c' \subset V$ does not exist: we can remove up to four vertices and still have a cycle, C, that visits every remaining vertex $v_i \in V$.

Note that a spider connects vertices $v_i, \ldots, v_{i+5}$. Thus, using spider edges, we can walk forward through the vertices in V, taking "step sizes" of up to five. We construct C by taking the next smallest available step forward, which will be to the next $v_i \notin V_c'$. The cut set, V_c', cannot block this path, since $|V_c'| < 5$. □

We now prove that adding the 5-spiders to G does not cause Laman's condition to be violated, or decrease the flexibility of G.

Lemma 6 *If a graph $G = (V, E)$ satisfies Laman's condition, G plus a 5-spider G_s satisfies Laman's condition.*

Proof. We examine all subgraphs of $G \cup G_s$ to verify that part 2 of Laman's condition holds. We know that all subgraphs of G satisfy part 2, and can easily verify that subgraphs of G_s satisfy part 2. Consider an induced subgraph on S, a subset of the spider body vertices, and $V \subset V$. If $|S| \geq 3$ and $|V'| \geq 3$, we know:

$$\begin{aligned} |E(S)| &\leq 3|S| - 6, \\ |E(V)| &\leq 3|V| - 6, \end{aligned}$$

and hence,

$$|E(S)| + |E(V)| \leq 3(|S| + |V|) - 12.$$

Since there are no more than six spider legs connecting the vertices in S and V, the subgraph satisfies part 2. Cases where $|S| < 3$ or $|V| < 3$ also satisfy part 2 of Laman's condition: we either add one vertex and up to two edges, or two vertices and up to five edges; either way, the inequality of part 2 holds. □

Lemma 7 *Adding a 5-spider, G_s, to a graph G cannot decrease the space of infinitesimal motions.*

Proof. We sketch the proof, using the terminology of Whiteley's survey [5] and Graver *et al.* [3].

Given a generic embedding of G, an infinitesimal motion of G is an assignment of "velocities" to the embedded vertices of G such that $R[v'_i]^T = \mathbf{0}$, where R is the rigidity matrix determined by the coordinates of the vertices. Each row in R represents an edge in G, and each column represents a coordinate of a vertex in G.

Let $G' = G \cup G_s$, and let the rigidity matrices of G, G', the spider body, and the spider legs be R, R', R_b, and $[R_{Lf} R_{Lb}]$, respectively. Then R' can be decomposed into:

$$R' = \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & R_b \\ R_{Lf} & R_{Lb} \end{bmatrix}$$

We know $R[v'_i]^T = \mathbf{0}$, and thus we need only find a solution to:

$$\begin{bmatrix} R_b \\ R_{Lb} \end{bmatrix} [v'_{bi}]^T = \begin{bmatrix} \mathbf{0} \\ -R_{Lf}[v'_i]^T \end{bmatrix}$$

where $[v'_{bi}]$ are the velocities of the spider body vertices. We know the rank of the above rigidity matrix is less than or equal to 18 since there are only 18 rows: 12 for the spider body edges, and 6 for the legs. Thus, there will be at least one valid assignment for the 6×3 coordinates of $[v'_{bi}]$. Therefore, for every valid infinitesimal motion of G, there exists a valid infinitesimal motion of G_s. $\square$

Theorem 8 *Any graph, G, can be embedded as a vertex-induced subgraph of a 5-vertex-connected graph, G_5, such that the space of infinitesimal motions is the same or larger, and G_5 satisfies Laman's condition if G does.*

Proof. By Lemma 5, we can use 5-spiders to increase the vertex connectivity of G to five. By Lemma 7, adding a 5-spider does not decrease the internal degrees of freedom of G, and G_5 retains the flexibility of G. By Lemma 6, G_5 satisfies Laman's condition if G does. $\square$

7 Conclusions

As we have shown in Sections 3, 4 and 5, we can increase the vertex connectivity of the double banana graph using spiders. In Section 6, we prove that any flexible graph satisfying Laman's condition is an induced subgraph of a 5-connected, flexible graph satisfying Laman's condition. Thus, we cannot use connectivity to create a necessary and sufficient test for combinatorial rigidity in 3D.

8 Open Problems

Problem 1 *Are the graphs in Figures 2(a) through (c) the smallest 3-, 4-, and 5-connected counterexamples to the sufficiency of Laman's condition?*

Problem 2 *Find a necessary and sufficient extension of Laman's condition to 3D.*

9 Acknowledgements

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