The Fitting Line Problem in the Laguerre Geometry

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Abstract

We address the problem of computing the fitting line of a set of circles in the Laguerre metric, that minimizes the distance to the farthest circle. To solve the fitting line problem we introduce a generalization of the concept of the width of a set of points using the Laguerre metric. We also present an efficient algorithm for finding the fitting line of a set of circles using minimization diagrams with running time $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.

1 Introduction

The Laguerre geometry is a geometry of the oriented spheres, originated from the space of generalized spheres [5, Section 20.6.4, p. 137]. In the Laguerre geometry, circles and lines are oriented: two circles have at most two common tangents. The classical problems in the Laguerre geometry that have been studied in computational geometry are (see [7]): the point inclusion in a union of circles, finding the connected components of a set of circles, and finding the contour of the union of circles. These problems have been successfully handled by using the Laguerre diagram, that is the generalization of the Voronoi diagram to the Laguerre geometry (see [7]).

In this paper, we present an algorithm for the determination of the fitting line of a set of circles in the Laguerre metric, that minimizes the distance to the farthest circle. The definition of the fitting line in the Laguerre geometry differs from the definition in the Euclidean geometry in the following way: the distance that is being minimized is the maximum Laguerre distance from the projection of a circle centre on the axis to the circle. The Laguerre metric between a point and a circle is the square root of the power of the point relatively to the circle. The concept of power, is intrinsically connected to the inversive geometry, and to the space of generalized spheres (see [4, Section 10.7.10 p. 175]. While in the Euclidean geometry, there exists a relationship between the width of a set of points and their convex hull, we will see that this relationship needs some care in the case of the Laguerre geometry. From this problem of the fitting line in the Laguerre geometry, we introduce a generalization of the concept of the minimal width of a set of points in the Laguerre metric. In order to introduce our generalization of the width in the Laguerre geometry, we will explore all the possible generalization alternatives, and see which one relates to the problem of the fitting line.

The width of a set of circles has a practical application in robotics: the length of the arm of a robot that would pick circular objects along a linear trajectory.

2 Preliminaries

Let $\mathbb{R}^2$ be the Euclidean plane by $\mathbb{R}^2$ and let $\mathbb{R}^3$ be the three dimensional vector space where the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is defined by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2}$ provided that $(x_1 - x_2)^2 + (y_1 - y_2)^2 \geq (z_1 - z_2)^2$. In the Laguerre geometry, a point $p\,(x, y, z)$ in $\mathbb{R}^3$ is mapped to a directed circle c in the Euclidean plane $\mathbb{R}^2$ with the centre (x, y), radius $|z|$, and direction corresponding to the sign of z. Let $\pi : \mathbb{R}^3 \to \mathbb{L}$ denote this map $\pi(p) = c$. The *Laguerre distance* between two points of $\mathbb{R}^3$ corresponds to the length of the common tangent (defined by the respective directions) of the corresponding two circles. The *Laguerre distance* from a point and a circle whose interior does not contain that point is the square root of the power of the point with respect to the circle. Let m be a point of $\mathbb{R}^2$. Let $S\,(a, r)$ denote the circle with centre a and the radius r. Its equation is given by $S\,(x) = 0$ where $S\,(x) = d\,(a, x)^2 - r^2$ (see [6]). The power of the point m with respect to the circle S is defined as [6]: $S\,(m) = d\,(a, m)^2 - r^2$. If $S\,(m) = 0$, the point m is on $S\,(a, r)$, if $S\,(m) < 0$, the point m is inside $S\,(a, r)$, if $S\,(m) > 0$, m is outside $S\,(a, r)$. If we take any line l passing through m and intersecting $S\,(a, r)$, at the intersection points t and t', we have the following property [6]: $\overrightarrow{mt} \cdot \overrightarrow{mt'} = S\,(m)$. Now consider two circles $S\,(a, r)$ and $S'\,(a', r')$. The locus of points whose power with respect to $S\,(a, r)$ equals their power with respect to $S'\,(a', r')$ is called the radical axis or *chordale* of $S\,(a, r)$ and $S'\,(a', r')$. This locus is defined if, and only if $a \neq a'$ (see Berger [4, Section 10.7.10.1, p. 175]). In the case where it is defined, its equation is: $S\,(x) = S'\,(x)$, and it is orthogonal to $\overrightarrow{aa'}$ [4, Section 10.7.10.1, p. 175].

The Laguerre geometry in $\mathbb{R}^3$ is actually a generalization of Euclidean geometry in $\mathbb{R}^2$ ($z = 0$). In this paper we generalize the notion of the (Euclidean) width of a set of points and introduce a new notion of *Laguerre width*. There are a few equivalent definitions of the Euclidean width [3]. One definition of the width of a set $A \subset \mathbb{R}^2$ is the minimum distance between two parallel supporting lines. One possible

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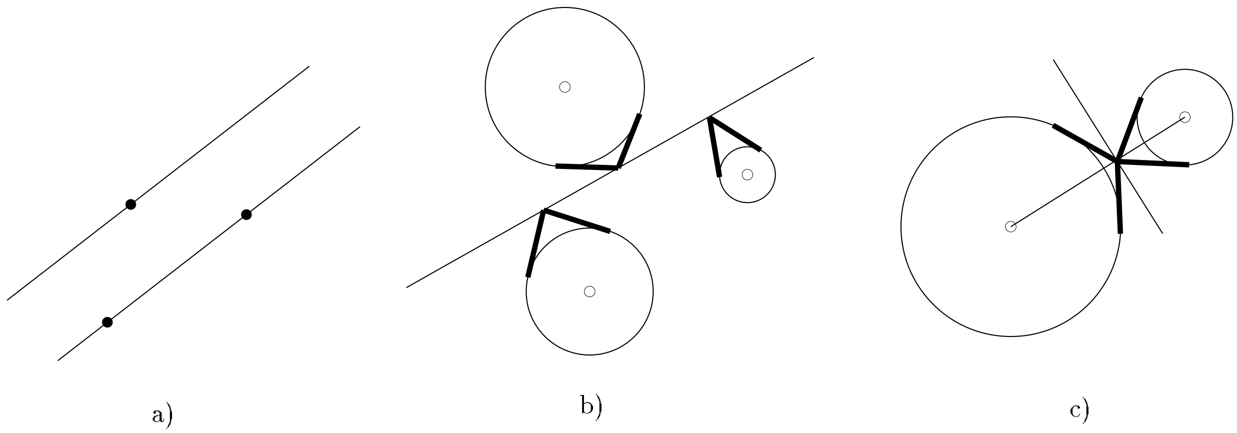

Figure 1: a) Euclidean width. b,c) Laguerre fitting line.

way to define Laguerre width of a set $B \subset \mathbb{R}^3$ is to take the minimum distance between two parallel supporting lines of a set of circles $\pi(B)$. It does not lead to a new notion because it corresponds to the Euclidean width of the union of circles $\pi(B)$.

Another definition of the Euclidean width can be expressed in terms of the *fitting line* in the plane. For a set A and a line l the *fitting distance* $fit(l)$ is defined as the largest distance from a point of A to its nearest point on l. The fitting line for a set $A \subset \mathbb{R}^2$ is defined as a line l minimizing fitting distance. The width of A is the double of fitting distance, see Fig 1. This applies to the Euclidean case as well as to the Laguerre case:

Definition 1 *We define the* Laguerre fit distance *between a circle c in $\mathbb{R}^2$ and a line l as the Laguerre distance between the point on l closest to the centre of c and c. We define the* Laguerre fitting distance *between a set C of directed circles in $\mathbb{R}^2$ and a line l as the largest Laguerre fit distance from a circle $c \in C$ and l. The* Laguerre fitting line *is defined as a line minimizing the Laguerre fitting distance. In $\mathbb{R}^3$ the Laguerre fitting line of a set of points $B \in \mathbb{R}^3$ can be viewed as a line l in the plane $z = 0$ minimizing the largest Laguerre distance from a point of B to its nearest point on the line l. The* Laguerre width *of a set C is the double of Laguerre fitting distance between C and the Laguerre fitting line, see Fig 1.*

The direction of the Euclidean fitting line of n points is defined by a pair of points, see Fig. 1 a). It is not longer valid for the Laguerre fitting line. The following lemma establishes the analogous property for the definition of the Laguerre fitting line.

Lemma 2 *The Laguerre fitting line l of n circles in the plane is defined by either 3 or 2 circles at largest distance from l. In both cases the fitting line is equidistant from these circles in Laguerre geometry. The Laguerre fitting line defined by 2 circles corresponds to the chordale of these two circles.*

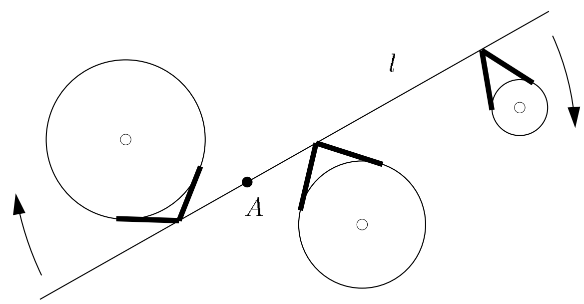

Figure 2: Rotation of l around A in clockwise order.

Proof. The cases of 2 and 3 circles are illustrated in Fig. 1 b,c). Let l be the Laguerre fitting line of n circles. If all circles at the largest distance D from l are on the same side of l then D can be improved by translating the line l towards these circles. By rotating the line l we can also reduce the distance D if the nearest points on L for the circles at distance D form two disjoint sequences corresponding to the sides of l, see Fig. 2. Otherwise there are either 2 or 3 circles equidistant from l, see in Fig. 1 b,c). $\square$

3 Computing the Laguerre Width

The brute force approach using Lemma 2 leads to the algorithm for computing the width of a set of points in the Laguerre geometry with $O(n^4)$ running time. Indeed, there are $O(n^2)$ pairs and $O(n^3)$ triples of points defining the fitting line. Each pair of points generates one possible fitting line and each triple of points generates at most 3 fitting line. To check each candidate line the algorithm computes all the distances from the circles to the line in linear time.

In this section we present two algorithms for computing the Laguerre width and the Laguerre fitting line. The first algorithm is based on the technique of rotating calipers and, for this, we introduce a generalization of the notion of the convex hull to the Laguerre geometry. The second algorithm is based on lower envelopes of bivariate functions and has better running time (although it is less practical).

3.1 Rotating Calipers Algorithm

Definition 3 *(Laguerre convex hull) The Laguerre convex hull $LCH(\mathcal{S})$ of a set $\mathcal{S} \subset \mathbb{L}$ is the smallest subset of $\mathbb{L}$ such as any linear combination of two elements S_i and S_j of $LCH(\mathcal{S})$ belongs to $LCH(\mathcal{S})$.*

This is equivalent to saying that $\pi^{-1}(LCH(\mathcal{S})) = CH(\pi^{-1}(\mathcal{S}))$, where CH denotes the traditional convex hull in $\mathbb{R}^3$, or the Laguerre convex hull of $\mathcal{S}$ is the image of the convex hull of the point set $B \subset \mathbb{R}^3$ such as $\pi(B) = \mathcal{S}$ by π.

This convex hull is particularly useful for computing the Laguerre width. We use the prefix δ to denote the boundary of a set.

Lemma 4 *The Laguerre fitting line of a set $\mathcal{S} \subset \mathbb{L}$ is defined by circles of $\pi\left(\delta CH\left(\pi^{-1}(\mathcal{S})\right)\right) \cap \mathcal{S}$.*

Proof. We will prove it by contradiction. Assume that one of the circles (say c) determining the fitting line l of a set $\mathcal{S} \subset \mathbb{L}$ does not belong to $\pi\left(\delta CH\left(\pi^{-1}(\mathcal{S})\right)\right) \cap \mathcal{S}$. Let $R = \pi^{-1}(c)$. Our first assumption is equivalent to saying that R is in the interior of $CH\left(\pi^{-1}(\mathcal{S})\right)$. Let L be the line of the plane of equation $z = 0$ in $\mathbb{R}^3$ whose equation in this plane is the same as l equation. In $\mathbb{R}^3$, the locus of points that are at the fitting distance d from the line L is a quadric D of axis of symmetry L. Since c defines the fitting line, R belongs to D. Let F be the closest point on the vertical plane containing L from R. Let X be the intersection point of the ray from R in the direction of the straight line (FR) away from F and $CH\left(\pi^{-1}(\mathcal{S})\right)$ (see Figure 3). Since R is in the interior of $CH\left(\pi^{-1}(\mathcal{S})\right)$ and on D, X and the vertical plane containing L lie on opposite sides with respect to D. Thus, the Laguerre distance from X to the point of L closest to X is greater than d. Consider now the polygonal facet P of $CH\left(\pi^{-1}(\mathcal{S})\right)$ which contains X. At least one vertex (say Q) of P lies on the same side as X with respect to D. Thus the Laguerre distance from Q to the point of L closest to Q is larger than d. But Q is a vertex of $\delta CH\left(\pi^{-1}(\mathcal{S})\right) \cap \pi^{-1}(\mathcal{S})$. Thus the fit distance of $\pi(Q)$ is greater than the fitting distance, a contradiction. □

We propose the following algorithm for computing the Laguerre width.

1. Map the circles to the Laguerre geometry and compute the convex hull of the corresponding points. Remove the points in interior of the convex hull. Remove the corresponding circles.
2. Apply the technique of rotating calipers. At every moment the algorithm maintains 2 or 3 circles defined the fitting line.

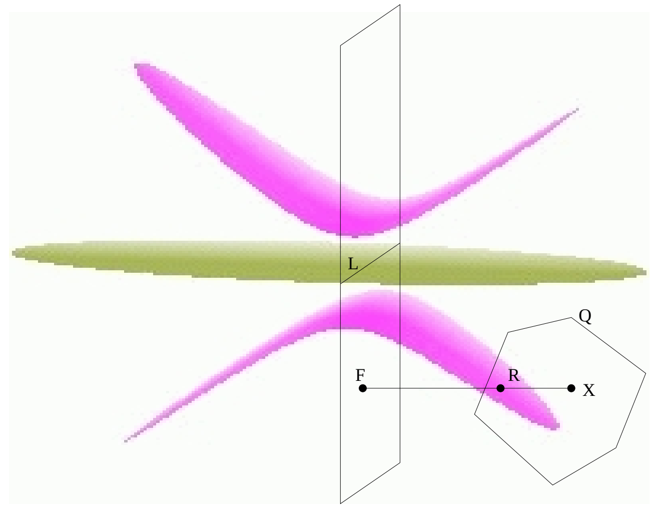

Figure 3: The proof of Lemma 4.

3. Compute the smallest fitting distance from rotating line to the circles from Step 2.

3.2 Algorithm Based on Lower Envelopes

Let w^* be the Laguerre width of the set $C = \{c_1, \ldots, c_n\}$ of n circles. Let $p_i(x_i, y_i)$ be the centre of c_i and r_i be the radius of c_i. We show how to compute w^* in $O(n^{2+\varepsilon})$ time. The algorithm finds a line minimizing the Laguerre fitting distance among (i) vertical lines and (ii) non-vertical lines. Then w^* is the smallest value among these two.

Vertical line. Since we can spend $O(n^{2+\varepsilon})$ time, we compute one number x^l for every pair of circles $(c_i, c_j), 1 \leq i < j \leq n$ such that the Laguerre distances from the line $x = x_l$ to c_i and to c_j are the same. There is one such value since x^l satisfies the following condition which is a linear equation in terms of x_l

$$(x^l - x_i)^2 - r_i^2 = (x^l - x_j)^2 - r_j^2.$$

We obtain at most $n(n-1)$ values of x^l (a pair (c_i, c_j) can be ignored if $x_i = x_j$ and $r_i = r_j$). Then we sort these values and traverse them in sorted order. We maintain the pair of circles on each side of the line $x = x^l$ that maximize the Laguerre distance to $x = x^l$. Every event can be processed in $O(1)$ time. The smallest value of the Laguerre fitting distance is achieved at one of the events.

Non-vertical line. We assume that the equation of l is $y = ax + b$ where $a, b \in \mathbb{R}$.

Lemma 5 *The projection of a point (c, d) onto a line $y = ax + b$ is the point with the following coordinates*

$$\left(\frac{c - ab + ad}{1 + a^2}, \frac{ac + b + a^2 d}{1 + a^2}\right). \tag{1}$$

Proof. Let (x', y') be the point from (1). It suffices to verify that (i) $y' = ax' + b$ and (ii) the vector $(x' - c, y' - d)$ is perpendicular to the vector $(1, a)$. The first condition follows from

$$\begin{aligned} ax' + b &= a\frac{c - ab + ad}{1 + a^2} + b \\ &= \frac{ac - a^2b + a^2d + b + a^2b}{1 + a^2} = y'. \end{aligned}$$

The second condition follows from

$$\begin{aligned} x' - c + a(y' - d) &= \frac{c - ab + ad}{1 + a^2} - c \\ &+ a(\frac{ac + b + a^2d}{1 + a^2} - d) = 0. \end{aligned}$$

□

Let $p'_i(x'_i, y'_i)$ be the projection of $p_i, i = 1, \ldots, n$ onto the line l. By Lemma 5

$$x'_i = \frac{x_i - ab + ay_i}{1 + a^2} \text{ and } y'_i = \frac{ax_i + b + a^2y_i}{1 + a^2}.$$

The Laguerre fitting distance between c_i and l is equal to

$$w_i = \sqrt{\max(0, (x' - x_i)^2 + (y' - y_i)^2 - r_i^2)}.$$

Thus $w_i \leq w$ is equivalent to

$$(x' - x_i)^2 + (y' - y_i)^2 \leq r_i^2 + w^2.$$

Lower envelope of bivariate functions. Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a collection of n bivariate functions, all algebraic of some constant degree. The *lower envelope* $E_{\mathcal{F}}$ of $\mathcal{F}$ is defined as

$$E_{\mathcal{F}}(\mathbf{x}) = \min_i f_i(\mathbf{x}).$$

Theorem 6 (Agarwal *et al.* **[1])** *The lower envelope of a collection of n bivariate functions, all algebraic of some constant degree, can be computed, in an appropriate model of computation, by a deterministic divide-and-conquer algorithm, in $O(n^{2+\varepsilon})$ time, for any $\varepsilon > 0$, where the constant of proportionality depends on ε and the maximum algebraic degree of the given functions (and of their domain boundaries).*

We define n bivariate functions

$$f_i(a, b) = (x' - x_i)^2 + (y' - y_i)^2 - r_i^2, i = 1, \ldots, n.$$

We apply the algorithm from Theorem 6 to compute the lower envelope of the functions f_i. It produces a subdivision of the (a, b)-plane into $O(n^{2+\varepsilon})$ faces, edges and vertices. For a given face ϕ, there exists an index j such that, for any point $(a, b) \in \phi$, the minimum value $\min_i f_i(a, b)$ is achieved by $f_j(a, b)$. We find the maximum value $f_j(a, b), (a, b) \in \phi$. The maximum value over all faces, edges and vertices of the subdivision is w^2. Therefore we proved the following result.

Theorem 7 *The Laguerre width of a set of n circles in the plane can be computed in $O(n^{2+\varepsilon})$ time.*

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