On the number of line tangents to four triangles in three-dimensional space *

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Abstract. We establish upper and lower bounds on the number of connected components of lines tangent to four triangles in $\mathbb{R}^3$. We show that four triangles in $\mathbb{R}^3$ may admit at least 88 tangent lines, and at most 216 isolated tangent lines, or an infinity (this may happen if the lines supporting the sides of the triangles are not in general position). In the latter case, the tangent lines may form up to 216 connected components, at most 54 of which can be infinite. The bounds are likely to be too large, but we can strengthen them with additional hypotheses: for instance, if no four lines, each supporting an edge of a different triangle, lie on a common ruled quadric (possibly degenerate to a plane), then the number of tangents is always finite and at most 162; if the four triangles are disjoint, then this number is at most 210; and if both conditions are true, then the number of tangents is at most 156 (the lower bound 88 still applies).

1 Introduction

In this paper, we are interested in lines tangents to four triangles. Our interest in lines tangent to triangles, and generally to polytopes in $\mathbb{R}^3$, is motivated by visibility problems. In computer graphics and robotics, scenes are often represented as unions of not necessarily disjoint polygonal or polyhedral objects. The objects that can be seen in a particular direction from a moving viewpoint may change when the line of sight becomes tangent to one or more objects in the scene. Since the line of sight then becomes tangent to a subset of the edges of the polygons and polyhedra representing the scene, questions about lines tangent to four polygons arise very naturally in this context.

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Our results. By a triangle in $\mathbb{R}^3$, we understand the convex hull of three distinct points in $\mathbb{R}^3$. We are not discussing degenerate triangles which reduce to a segment or to a point. Given four triangles t_1, t_2, t_3, and t_4 in $\mathbb{R}^3$, denote by $n(t_1, t_2, t_3, t_4)$ the number of lines tangent to all four triangles.[1] Note that this number can be infinite if, for example, four sides of the triangles are supported by four lines that lie on a hyperbolic paraboloid. Let us denote by T_4 the set of all quadruplets of triangles (t_1, t_2, t_3, t_4) with the property that for any of the $3^4 = 81$ quadruplets of lines $(\ell_1, \ell_2, \ell_3, \ell_4)$ such that ℓ_i supports an edge of t_i, the four lines do not belong to a quadric (a paraboloid hyperbolic or a hyperboloid of one sheet), and no two of these lines are coplanar. In particular, for every $(t_1, t_2, t_3, t_4) \in T_4$, there are at most two lines tangent to the lines supporting any quadruplet of edges, hence $n(t_1, t_2, t_3, t_4)$ is finite and at most 162.

In this paper, we are primarily interested in the number

$$n_4^{\text{triangles}} = \max_{(t_1,t_2,t_3,t_4)\in T_4} n(t_1, t_2, t_3, t_4)$$

Our main results are two-fold. First, we show that

Theorem 1 *We have $n_4^{\text{triangles}} \geqslant 88$. More precisely, there is a configuration of four* disjoint *triangles in $\mathbb{R}^3$ which admit finitely many, but at least 88, distinct tangent lines.*

Next, we improve the upper bound on n_4 slightly, in the disjoint case.

Theorem 2 *We have $n_4^{\text{triangles}} \leqslant 162$. More precisely, if four triangles are in T_4, they admit at most 162 distinct tangent lines. This number is at most 156 if the triangles are disjoint.*

Unfortunately, we cannot claim that if the number of tangent lines is finite, then it is at most 162, because the number may be finite although the four triangles do not belong to T_4. When the four triangles are not in T_4, the number of lines tangent to all four triangles can be infinite, and even when it is finite it could be more than 162. In this case, we may group these tangents by connected components: two line tangents are in the same component if one may move continuously

[1] A line tangent to four triangles does not properly cross the interior of these triangles, and so it corresponds to an unoccluded line of sight. If it is contained in the plane of any of these triangles, it may intersect the interior but it is not considered a proper crossing. Indeed, the line is still tangent to the triangle considered as a degenerate three-dimensional polytope.

between the two lines while staying tangent to the four triangles. Let $n'(t_1, t_2, t_3, t_4)$ denote the number of *connected components* of tangent lines to four triangle, and let

$$n_4'^{\,\text{triangles}} = \max_{any\,(t_1,t_2,t_3,t_4)} n'(t_1, t_2, t_3, t_4)$$

Each quadruplet of edges may induce up to four components of tangent lines [2], bringing the upper bound to 324. We can give a better bound on the number n_4' of connected components of lines tangent to four triangles in any position. We only state the following theorem (the proof will appear in the complete version).

Theorem 3 *We have* $n_4'^{\,\text{triangles}} \leqslant 216$ *(and* 210 *if the triangles are disjoint). Moreover, the number of* infinite *components is bounded above by* 54*.*

2 Proof of Theorem 1

For the lower bound, we construct four disjoint triangles in such a way that they admit at least 88 tangents. At the heart of our construction is a perturbation scheme from a configuration of lines l_1, l_2, l_3 and l_4 which have exactly two transversal lines x and y. We will perturb each l_i into coplanar lines, l_i' and l_i'', in order to multiply x and y into two sets of tangent lines. By choosing the perturbation carefully, we argue that those tangent lines will be tangent to the triangles t_i defined by the three lines l_i, l_i', and l_i''.

One way to obtain such a configuration is by taking l_1, l_2, l_3 on a hyperbolic paraboloid (see Figure 1). This paraboloid admits two families of ruling lines, and we take l_1, l_2, l_3 in one of the two families. Next we choose a vertical plane π_4 intersecting the paraboloid in a conic $\mathcal{C}$ (actually, a parabola; see Figure 1) and a line l_4 in π_4 that cuts $\mathcal{C}$ in two points, x_4 and y_4. The lines that belong to the second family of lines ruling the paraboloid passing through these two points are denoted x and y, and and intersect $l_1, \ldots, l_4$. In order to avoid any kind of degenerate configurations, we may take all four lines algebraically independent.

For our construction, a bit of notation helps. Given three skew lines a, b, c, we denote by $\mathcal{L}(a, b, c)$ the set of their line transversals, and by $\mathcal{Q}(a, b, c)$ the quadric ruled by these lines. In particular we will denote by $\mathcal{Q}_j$ the quadric passing through the lines l_i for all $i \in \{1, 2, 3, 4\}$ distinct from j. We denote by π_i a (not necessarily vertical) plane passing through l_i ($i = 1, 2, 3, 4$). Note that each plane π_i intersects the corresponding quadric $\mathcal{Q}_i$ in a non-degenerate conic $\mathcal{C}_i$, and in this plane the line l_i intersects $\mathcal{C}_i$ in two points, $x_i = x \cap \pi_i$ and $y_i = y \cap \pi_i$. We can always pick π_i such that $\mathcal{C}_i$ is a parabola, or in case of a hyperbola, such that l_i intersects the same branch twice. This will be important in the construction below and is referred to as the *local convexity* of $\mathcal{C}_i$ in the neighborhood of x and y.

Construction of $\mathbf{t_4}$. The situation in π_4 is depicted in Figure 2(left). The first step of our construction is to pick

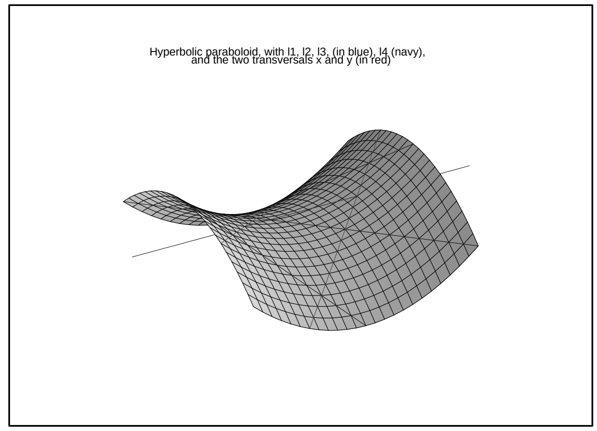

Figure 1: The initial configuration l_1, l_2, l_3 and l_4 with the hyperbolic paraboloid $\mathcal{Q}_4$.

a point on l_4 outside the conic $\mathcal{C}_4$ (on the side of x_4) and rotate l_4 into a line l_4' by a very small angle ε_4. This introduces two points x_4' and y_4'. Then we pick a line l_4'' which intersects $\mathcal{C}_4$ in two points in the very small arc from y_4 to y_4'. Note that this line is almost tangent to $\mathcal{C}_4$. The lines l_4, l_4' and l_4'' thus intersects $\mathcal{C}_4$ into six points, which are as close as we want to x_4 and y_4. The local convexity of $\mathcal{C}_4$ around y ensures that those points actually lie on the triangle t_4 bounded by l_4, l_4' and l_4''.[2] These six points corresponds to six lines that are transversal to l_1, l_2, l_3 and tangent to the triangle t_4, and which are as close as we want to x and y. (See Figure 2(right).)

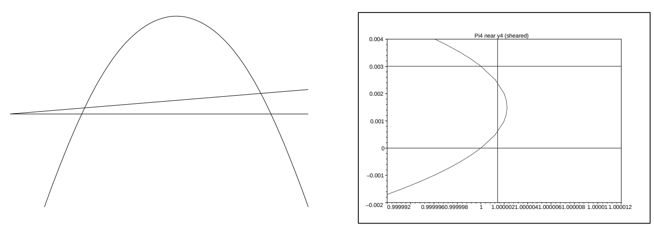

Figure 2: (left) In π_4, the line l_4 cuts $\mathcal{C}_4$ in two points, x_4 and y_4. (right) From 2 intersections to 6.

Construction of $\mathbf{t_3}$. The second step takes place in π_3. The quadric $\mathcal{Q}(l_1, l_2, l_4')$ cuts π_3 in a conic C_3' very close to $\mathcal{C}_3$, while $\mathcal{Q}(l_1, l_2, l_4'')$ cuts π_3 in a conic C_3'' (not necessarily close to $\mathcal{C}_3$). Note that $\mathcal{C}_3'$ intersects l_3 in two points x_3' and y_3' very close to x_3 and y_3, while $\mathcal{C}_3''$ intersects l_3 in two points between y_3 and y_3'. Thus either (i) C_3'' is almost tangent to l_3, or (i) it is hyperbola whose two branches are almost parallel in the neighborhood of y_3. (See Figure 3(left)).

In any case, we pick a point on l_3 outside the segment (x_3, y_3) (this time on the side of y_3) and rotate l_3 into a line l_3' by a small angle ε_3. Thus l_3' intersects $\mathcal{C}_3$ in two points close to x_3 and y_3 and $\mathcal{C}_3'$ in two points close to x_3' and y_3'.

[2] Local convexity is crucial here: If $\mathcal{C}_4$ had been concave in a neighborhood of y, as would have happened if $\mathcal{C}_4$ had been a hyperbola and l_4 had cut its two branches, then l_4'' would have actually put y_4 and y_4' outside the triangle t_4.

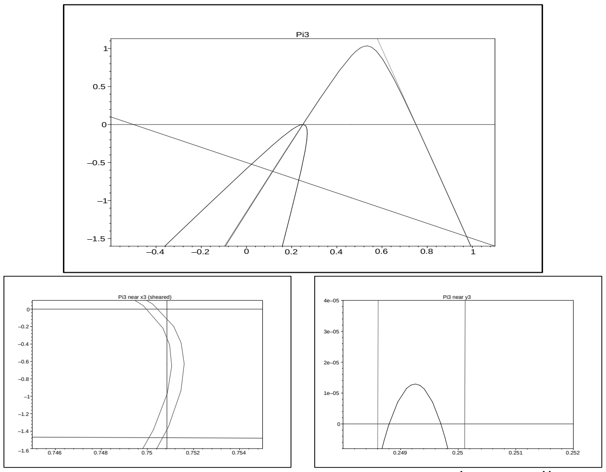

Figure 3: (top) In π_3, the line l_3 cuts $\mathcal{C}_3$, $\mathcal{C}'_3$ and $\mathcal{C}''_3$ in six points, close to x_3 and y_3. (bottom) From 6 intersections to $6+6+4=16$: (left) near x_3 (right) near y_3.

By choosing ε_3 small enough (ε_4 being fixed) we can also guarantee that l'_3 intersects $\mathcal{C}''_3$ in two points close to y_3 and y'_3. Finally, we choose ε_3 big enough with respect to the curvature of $\mathcal{C}_3$ and $\mathcal{C}'_3$ so that[3] the portions of $\mathcal{C}_3$ and $\mathcal{C}'_3$ close to x_3 and x'_3 in the angular sector between l_3 and l'_3 both admit a line l''_3 that intersects both conics in two points each within that sector. Note that l''_3 is almost tangent to both curves $\mathcal{C}_3$ and $\mathcal{C}'_3$.

Note the apparent contradiction: ε_3 must be big enough w.r.t. curvature of and distance between $\mathcal{C}_3$ and $\mathcal{C}'_3$ to allow for the existence of l''_3, yet small enough for l'_3 to intersect $\mathcal{C}''_3$. We resolve it by arguing that choosing the direction of rotation of l'_3 carefully: In case (i), we rotate l'_3 towards the direction of the concavity of $\mathcal{C}''_3$. Thus the two intersections with $\mathcal{C}''_3$ still exist for quite large values of ε_3. Note that case (ii) poses no problem. This essentially removes the contradiction.

Again, the local convexity of both $\mathcal{C}_3$ and $\mathcal{C}'_3$ is used to guarantee that all these points lie on the triangle t_3 bounded in π_3 by l_3, l'_3 and l''_3. Together, l_1, l_2, t_3 and t_4 have $6+6+4=16$ tangent lines. The situation is depicted in Figure 3(top).

Construction of $\mathbf{t_2}$. In π_2, in addition to $\mathcal{C}_2$, we now have three other conics very close to $\mathcal{C}_2$ (intersection with π_2 of[4] $\mathcal{Q}(l_1, l_3, l'_4)$, $\mathcal{Q}(l_1, l'_3, l_4)$, and $\mathcal{Q}(l_1, l'_3, l'_4)$). There are also a second group of two conics resulting from the intersection with π_2 of $\mathcal{Q}(l_1, \{l_3, l'_3\}, l''_4)$, which may be almost tangent to l_2 near y_2 as in case (i) above, or hyperbolas whose two branches intersect l_2 near y_2 as in case (ii) above. Similarly, there is a third group of two conics resulting from the intersection with π_2 of $\mathcal{Q}(l_1, l''_3, \{l_4, l'_4\})$, which intersect l_2 near x_2 (either case (i) or (ii)). (See Figure 4(left).)

[3]This is the sore point: ε_3 must be big enough w.r.t. curvature of and distance between $\mathcal{C}_3$ and $\mathcal{C}'_3$ to allow for l''_3, yet small enough for l'_3 to intersect $\mathcal{C}''_3$. Until we do the concrete construction, the doubt remains...

[4]We will extend $\mathcal{Q}()$ with a set-theoretic notation to avoid tedious repetitions. For instance, $\mathcal{Q}(l_1, \{l_3, l'_3\}, \{l_4, l'_4\})$ refers to the union of the four possible combinations.

As before, we pick a point on l_2 outside the segment (x_2, y_2) (say near y_2) and rotate l_2 into a line l'_2 by a small angle ε_2. Unfortunately, if the second and third groups are both in case (i) and their tangencies are on opposite sides of l_2, we cannot choose the direction of rotation as for l_3 above, because we may lose the intersections with the group whose tangency is on the other side of the direction of the rotation. It turns out that we can place the four lines l_1, l_2, l_3, and l_4 such that the second and third groups are both tangent to l_2 on the same side. Thus we can choose to rotate l'_2 towards that direction (without constraints on ε_2) and intersect the first group of conics in eight points, and the second and third groups in another eight points, four near y_2 and four near x_2, introducing sixteen new transversals.

As for l''_2, we choose it almost tangent to the first group of four conics so that intersects all four twice near x_2 in the angular sector between l_2 and l'_2. Again, the apparent contradiction on the order of magnitude of ε_2 w.r.t. the curvature of these conics near x_2 and the need for ε_2 to be small is resolved by the direction of rotation which guarantees the existence of the intersections between l'_2 and the second group of conics even for rather large values of ε_2. Thus l''_2 introduces an additional eight new transversals.

Let the triangle t_2 be bounded in π_2 by l_2, l'_2 and l''_2. Again, the local convexity of all the conics guarantees that all the new transversals to l_2, l'_2 and l''_2 are actually tangent to the triangle t_3 bounded in π_3 by l_3, l'_3 and l''_3. Together, l_1, t_2, t_3 and t_4 have $16+12+8=36$ tangent lines. (See Figure 4(right).)

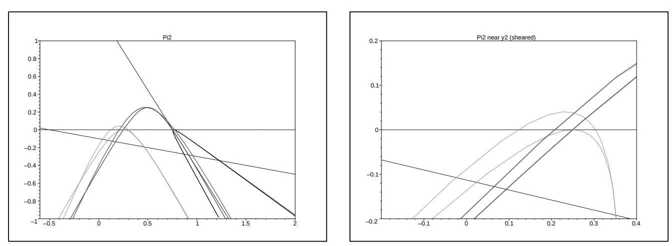

Figure 4: (left) In π_2, the line l_2 cuts three groups of conics, those close to $\mathcal{C}_2$, those tangent to l_2 at x_2, and those tangent at y_2. (right) From 16 intersections to $16+16+8=40$.

Construction of $\mathbf{t_1}$. In π_1, the situation has multiplied. Close to $\mathcal{C}_1$ are eight conics (including $\mathcal{C}_1$) intersection of π_1 with $\mathcal{Q}(\{l_2, l'_2\}, \{l_3, l'_3\}, \{l_4, l'_4\})$. There are also four conics (second group) intersecting l_1 near y_1, resulting from the quadrics $\mathcal{Q}(\{l_2, l'_2\}, \{l_3, l'_3\}, l_4, l''_4)$. And two groups (third and fourth) of four conics each, intersecting l_1 near x_1, which result from $\mathcal{Q}(l''_2, , \{l_3, l'_3\}, \{l_4, l'_4\})$ and $\mathcal{Q}(\{l_2, l'_2\}, l''_3, \{l_4, l'_4\})$. (See Figure 5(left).)

We play the same game, and rotate l_1 into l'_1 by an angle ε_1, introducing sixteen new transversals with the first group of conics. We cannot ignore the case where the second, third and fourth groups all fall in case (i), but in this case at least two groups share the same side of tangency, so we can choose the direction of rotation of l'_1 to introduce at least another sixteen new transversals, without restrictions on ε_1. Finally, we can choose l_1'' to close the triangle t_1 in such a way that its side cuts the eight conics of the first group between l_1 and l_1' into sixteen new points, all on the boundary of t_1 by again using the local convexity of all conics near x_1 and y_1. The situation is depicted in Figure 5(right).

Hence the four triangles thus constructed have a total of $40+16+16+16=88$ lines tangent, finishing the proof of Theorem 1.

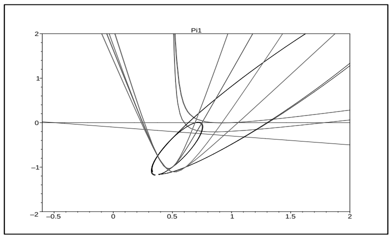

Figure 5: In π_1, the line l_1 cuts eight conics (first group), and three groups of four conics each, bringing the number of intersections from 36 to $40+16+16+16=88$.

Remark. In what precedes, we have only accounted for the tangents that pass through only one of the sides supported by l_1'', l_2'', l_3'', and l_4''. Because of the short length of each of these segments, it is hard to say whether there are common tangents to the triangles through more than one of these sides. If the construction could be more controlled, perhaps the lower bound could be increased.

3 Proof of Theorem 2

It is known that four segments have at most four transversals (or an infinity); moreover, if the four supporting lines do not belong to a common ruled surface, then there can be at most two transversals[2]. Thus if the triangles are in T_4, the four triangles have at most $3^4=81$ *quadruplets of edges* formed by picking an edge from each triangle. Each quadruplet can have at most two transversals, and hence we very easily obtain $n_4^{\text{triangles}} \leqslant 81 \times 2 = 162$.

We now indicate how to improve on this bound when the triangles are disjoint. We can show that there are at most 78 quadruplets to consider in the disjoint case, thus bounding the number of common tangents by 156. The proof follows that on the upper bound for the number of tangents to four polytopes[1], but limits the number of configurations for disjoint triangles in $\mathbb{R}^3$. For clarity, we divide the proof into two lemmas. For lack of space, however, we do not include the proofs of Lemma4, and only sketch the proof of Lemma 5.

Lemma 4 *Fix an edge e of a triangle, say t_1. The number of quadruplets of common tangents which contain e is always at most 27, at most 26 if the line supporting e stabs only one of the triangles t_2, t_3 or t_4, and at most 25 if it stabs none. Those bounds are tight.*

Lemma 5 *Given four disjoint triangles, the number of quadruplets that lead to a common tangent is bounded by 78.*

Proof. (Sketch) The proof proceeds by constructing a bipartite graph between twelve nodes representing each edge e_i^j of every triangle t_j ($i=1,2,3$ and $j=1,2,3,4$) and four nodes representing each triangle t_k ($k \neq j$). An arc between e_i^j and t_k indicates that the line supporting e_i^j stabs t_k. (We use *arc* to describe the edges of the graph, in order to avoid confusion between edges of the graph and edges of the triangles.) The proof rests on the claim that this graph can have at most 18 edges (out of a possible 48). We do not prove the claim for lack of space, but its proof rests on a careful examination of the relative position of two disjoint triangles, and using Lemma 4. □

Remark. In the disjoint case, it is possible to pick four triangles whose bipartite graph has exactly 18 edges, showing that the argument above cannot be improved further without additional ideas. It is conceivable, however, that finding further restrictions on the bipartite graph may lead to lower the upper bound.

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References

[1] H. Brönnimann, O. Devillers, V. Dujmovic, H. Everett, M. Glisse, X. Goaoc, S. Lazard, H.-S. Na, and S. Whitesides. On the number of lines tangent to four convex polyhedra. *Proc. Fourteenth Canad. Conf. Comput. Geom.*, pp. 113-117, 2002.

[2] H. Brönnimann, H. Everett, F. Sottile, S. Lazard, and S. Whitesides. Transversals to line segments in $\mathbb{R}^3$. *Proc. Fifteenth Canad. Conf. Comput. Geom.*, 2003.

[3] F. Durand, G. Drettakis, and C. Puech. The 3D visibility complex. *ACM Trans. Graphics* 21(2):176–206, 2002.

[4] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Heidelberg, Germany, 1987.

[5] J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In *New Trends in Discrete and Computational Geometry*, (J. Pach, ed.), Springer Verlag, Heidelberg, pp. 163-198, 1993.

[6] H. Pottman and J. Wallner. *Computational Line Geometry*. Springer-Verlag, Berlin, 2001.

[7] R. Wenger. Progress in geometric transversal theory. In *Advances in Discrete and Computational Geometry*, (B. Chazelle, J. E. Goodman, and R. Pollack, eds.), Amer. Math. Soc., Providence, pp. 375–393, 1998.