Three-Dimensional 1-Bend Graph Drawings*

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Abstract

We consider three-dimensional grid-drawings of graphs with at most one bend per edge. Under the additional requirement that the vertices be collinear, we prove that the minimum volume of such a drawing is $\Theta(cn)$, where n is the number of vertices and c is the cutwidth of the graph. We then prove that every graph has a three-dimensional grid-drawing with $\mathcal{O}(n^3/\log^2 n)$ volume and one bend per edge. The best previous bound was $\mathcal{O}(n^3)$.

1 Introduction

We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. A *three-dimensional polyline grid-drawing* of a graph, henceforth called a *polyline drawing*, represents the vertices by distinct points in $\mathbb{Z}^3$ (called *gridpoints*), and represents each edge as a polyline between its endpoints with bends (if any) also at gridpoints, such that distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. A polyline drawing with at most b bends per edge is called a *b-bend drawing*. A 0-bend drawing is called a *straight-line drawing*.

A folklore result states that every graph has a straight-line drawing. Thus we are interested in optimising measures of the aesthetic quality of such drawings. The *bounding box* of a polyline drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X-1$, $Y-1$ and $Z-1$, then we speak of an $X \times Y \times Z$ polyline drawing with *volume* $X \cdot Y \cdot Z$. That is, the volume of a polyline drawing is the number of gridpoints in the bounding box.

This paper continues the study of upper bounds on the volume and number of bends per edge in polyline drawings. The volume of straight-line drawings has been widely studied (see [6]). Only recently have (non-orthogonal) polyline drawings been considered [4, 8]. Table 1 summarises the best known upper bounds on the volume and bends per edge in polyline drawings.

Cohen *et al.* [2] proved that the complete graph K_n (and hence every n-vertex graph) has a straight-line drawing with $\mathcal{O}(n^3)$ volume, and that $\Omega(n^3)$ volume was necessary. Dyck *et al.* [8] recently proved that K_n has a 2-bend drawing with $\mathcal{O}(n^2)$ volume. The same conclusion can be reached from the $\mathcal{O}(qn)$ volume bound of Dujmović and Wood [4], since trivially every graph has a $(n-1)$-queue layout. Dyck *et al.* [8] asked the interesting question: what is the minimum volume in a 1-bend drawing of K_n? The best known upper bound at the time was $\mathcal{O}(n^3)$, while $\Omega(n^2)$ is the best known lower bound. (Bose *et al.* [1] proved that all polyline drawings have $\Omega(n+m)$ volume.)

In this paper we prove two results. The first concerns *collinear* polyline drawings in which all the vertices are in a single line. Let G be a graph, and let σ be a linear order of $V(G)$. Let $L_\sigma(e)$ and $R_\sigma(e)$ denote the endpoints of each edge e such that $L_\sigma(e) <_\sigma R_\sigma(e)$. For each vertex $v \in V(G)$, the set $\{e \in E(G) : L_\sigma(e) \leq_\sigma v <_\sigma R_\sigma(e)\}$ is called the *cut* in σ at v. The *cutwidth* of σ is the maximum size of a cut in σ. The *cutwidth* of G is the minimum cutwidth of a linear order of $V(G)$.

Theorem 1 *Let G be a graph with n vertices and cutwidth c. The minimum volume for a 1-bend collinear drawing of G is $\Theta(cn)$.*

Theorem 1 represents a qualitative improvement over the $\mathcal{O}(nm)$ volume bound of Dujmović and Wood [4]. Our second result improves the best known upper bound for 1-bend drawings of K_n.

Theorem 2 *Every complete graph K_n, and hence every n-vertex graph, has a 1-bend $\mathcal{O}(\log n) \times \mathcal{O}(n) \times \mathcal{O}(n^2/\log^3 n)$ drawing with $\mathcal{O}(n^3/\log^2 n)$ volume.*

It is not straightforward to compare the volume bound in Theorem 2 with the $\mathcal{O}(kqm)$ bound by Dujmović and Wood [4] for k-colourable q-queue graphs (see Table 1). However, since $k \leq 4q$ and $m \leq 2qn$ (see [7]), we have that $\mathcal{O}(kqm) \in \mathcal{O}(q^3 n)$, and thus the $\mathcal{O}(kqm)$ bound by Dujmović and Wood [4] is no more than the bound in Theorem 2 whenever the graph has a $\mathcal{O}((n/\log n)^{2/3})$-queue layout. On the other hand, $kqm \geq m^2/n$. So for dense graphs with $\Omega(n^2)$ edges the $\mathcal{O}(kqm)$ bound by Dujmović and Wood [4] is cubic (in n), and the bound in Theorem 2 is necessarily smaller. In particular, Theorem 2 provides a partial solution to the above-mentioned open problem of Dyck *et al.* [8] regarding the minimum volume of a 1-bend drawing of K_n.

2 Proof of Theorem 1

First we prove the lower bound in Theorem 1.

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Table 1: Volume of 3D polyline drawings of graphs with n vertices and $m \geq n$ edges.

graph family	bends per edge	volume	reference
arbitrary	0	$\mathcal{O}(n^3)$	Cohen *et al.* [2]
arbitrary	0	$\mathcal{O}(m^{4/3}n)$	Dujmović and Wood [6]
maximum degree Δ	0	$\mathcal{O}(\Delta mn)$	Dujmović and Wood [6]
bounded chromatic number	0	$\mathcal{O}(n^2)$	Pach, Thiele and Tóth [9]
bounded chromatic number	0	$\mathcal{O}(m^{2/3}n)$	Dujmović and Wood [6]
bounded maximum degree	0	$\mathcal{O}(n^{3/2})$	Dujmović and Wood [6]
H-minor free (H fixed)	0	$\mathcal{O}(n^{3/2})$	Dujmović and Wood [6]
bounded tree-width	0	$\mathcal{O}(n)$	Dujmović and Wood [5]
k-colourable q-queue	1	$\mathcal{O}(kqm)$	Dujmović and Wood [4]
arbitrary	1	$\mathcal{O}(nm)$	Dujmović and Wood [4]
cutwidth c	1	$\mathcal{O}(cn)$	Theorem 1
arbitrary	1	$\mathcal{O}(n^3/\log^2 n)$	Theorem 2
q-queue	2	$\mathcal{O}(qn)$	Dujmović and Wood [4]
q-queue (constant $\epsilon > 0$)	$\mathcal{O}(1)$	$\mathcal{O}(mq^\epsilon)$	Dujmović and Wood [4]
q-queue	$\mathcal{O}(\log q)$	$\mathcal{O}(m \log q)$	Dujmović and Wood [4]

Lemma 3 *Let G be a graph with n vertices and cutwidth c. Then every 1-bend collinear drawing of G has at least $cn/2$ volume.*

Proof. Consider a 1-bend collinear drawing of G in an $X \times Y \times Z$ bounding box. Let L be the line containing the vertices. If L is not contained in a grid-plane, then $X, Y, Z \geq n$, and the volume is at least $n^3 \geq cn$.

Now assume, without loss of generality, that L is contained in the $Z = 0$ plane. Let σ be a linear order of the vertices determined by L. Let B be the set of bends corresponding to the edges in the largest cut in σ. Then $|B| \geq c$. For every line L' parallel to L, there is at most one bend in B on L', as otherwise there is a crossing.

First suppose that L is axis-parallel. Without loss of generality, L is the X-axis. Then $X \geq n$. The gridpoints in the bounding box can be covered by YZ lines parallel to L. Thus $YZ \geq |B| \geq c$, and the volume $XYZ \geq cn$.

Now suppose that L is not axis-parallel. Thus $X \geq n$ and $Y \geq n$. The gridpoints in the bounding box can be covered by $Z(X+Y)$ lines parallel to L. Thus $Z(X+Y) \geq |B| \geq c$, and the volume $XYZ \geq XY\, c/(X+Y) \geq cn/2$. □

To prove the upper bound in Theorem 1 we will need the following lemma, which is a slight generalisation of a well known result. (For example, Pach, Thiele and Tóth [9] proved the case $X = Y$). We say two gridpoints v and w in the plane are *visible* if the segment vw contains no other gridpoint.

Lemma 4 *The number of gridpoints $\{(x, y) : 1 \leq x \leq X, 1 \leq y \leq Y\}$ that are visible from the origin is at least $3XY/2\pi^2$.*

Proof. Without loss of generality $X \leq Y$. Let N be the desired number of gridpoints. For each $1 \leq x \leq X$, let N_x be the number of gridpoints (x, y) that are visible from the origin, such that $1 \leq y \leq Y$. A gridpoint (x, y) is visible from the origin if and only if x and y are coprime. Let $\phi(x)$ be the number of positive integers less than x that are coprime with x (Euler's ϕ function). Thus $N_x \geq \phi(x)$, and

$$N = \sum_{x=1}^{X} N_x \geq \sum_{x=1}^{X} \phi(x) \approx \frac{3X^2}{\pi^2} .$$

If $X \geq Y/2$, then $N \geq 3XY/2\pi^2$, and we are done. Now assume that $Y \geq 2X$. If x and y are coprime, then x and $y + x$ are coprime. Thus $N_x \geq \lfloor Y/x \rfloor \cdot \phi(x)$. Thus,

$$N \geq \sum_{x=1}^{X} \left\lfloor \frac{Y}{x} \right\rfloor \cdot \phi(x) \geq \left(\frac{Y - X}{X} \right) \sum_{x=1}^{X} \phi(x) \approx \frac{3(Y-X)X}{\pi^2} \geq \frac{3XY}{2\pi^2}$$

□

Now we prove the following strengthening of the upper bound in Theorem 1.

Lemma 5 *Let G be a graph with n vertices and cutwidth c. For all integers $X \geq 1$, G has a 1-bend collinear $X \times \mathcal{O}(c/X) \times n$ drawing with the vertices on the Z-axis. The volume is $\mathcal{O}(cn)$.*

Proof. Let σ be a vertex ordering of G with cutwidth c. For all pairs of distinct edges e and f, say $e \prec f$ whenever $R_\sigma(e) \leq_\sigma L_\sigma(f)$. Then $\preceq$ is a partial order on $E(G)$, where an antichain in $\preceq$ is a cut in σ. By Dilworth's Theorem [3], there is a partition of $E(G)$ into chains $E_1, E_2, \ldots, E_c$, such that each $E_i = (e_{i,1}, e_{i,2}, \ldots, e_{i,k_i})$ and $R_\sigma(e_{i,j}) \leq_\sigma L_\sigma(e_{i,j+1})$ for all $1 \leq j \leq k_i - 1$.

By Lemma 4 with $Y = \lceil 4\pi^2 c/3X \rceil$, there is a set $S = \{(x_i, y_i) : 1 \leq i \leq c, 1 \leq x_i \leq X, 1 \leq y_i \leq Y\}$ of gridpoints that are visible from the origin. Position the ith vertex in σ at $(0, 0, i)$ on the Z-axis, and position the bend for each edge $e_{i,j}$ at (x_i, y_i, j). Edges in distinct chains are contained in distinct planes that only intersect in the Z-axis. Thus such edges do not cross. Edges within each chain E_i do not cross since no two edges in E_i are nested or crossing in σ, and the Z-coordinates of the bends of the edges in E_i agrees with the order of their endpoints on the Z-axis, as illustrated in Figure 1. The bounding box is $X \times \lceil 4\pi^2 c/3X \rceil \times n$, since the number of edges in a single chain is at most $n - 1$. $\square$

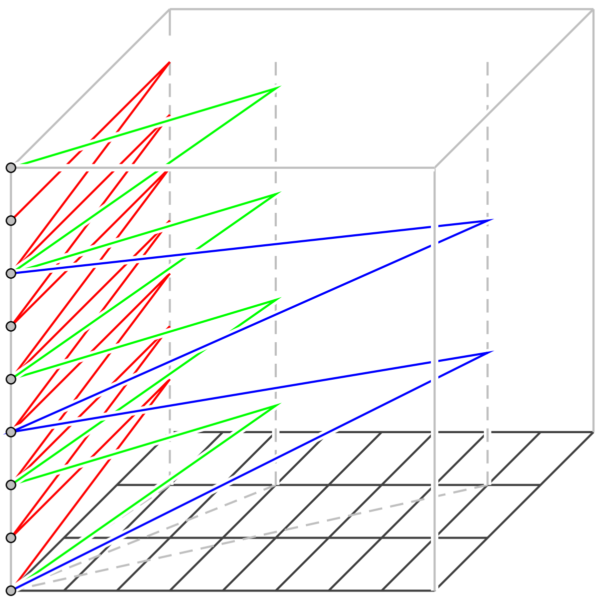

Figure 1: Construction in Lemma 5.

The constants in Lemma 5 can be tweaked as follows.

Lemma 6 *Let G be a graph with n vertices and cutwidth c. Then G has a 1-bend collinear $3 \times \lceil \frac{c-2}{2} \rceil \times n$ drawing. The volume is at most $3(c-1)n/2$.*

Proof. Let $S = \{(-1, 0), (1, 0)\} \cup \{(x, 1), (x, -1) : -1 \leq x \leq \lceil (c-6)/2 \rceil\}$. Then S consists of at least c gridpoints that are visible from the origin. The result follows from the proof of Lemma 5. $\square$

Since the cutwidth of K_n is $n^2/4$ we have:

Corollary 7 *The minimum volume for a 1-bend collinear drawing of the complete graph K_n is $\Theta(n^3)$. For all $X \geq 1$, K_n has a 1-bend collinear $X \times \mathcal{O}(n^2/X) \times n$ drawing with the vertices on the Z-axis. Furthermore, K_n has a 1-bend collinear $3 \times \lceil n^2/8 \rceil \times n$ drawing with volume at most $3n^3/8$.* $\square$

3 Proof of Theorem 2

Let $P = \lceil \frac{1}{2} \log_4 n \rceil$ and $Q = \lceil n/P \rceil$. Let $V(K_n) = \{v_{a,i} : 1 \leq a \leq P, 1 \leq i \leq Q\}$. Position each vertex $v_{a,i}$ at

$$(2a,\ aQ + i,\ 0)\ .$$

For each $1 \leq a \leq P$, the set of vertices $\{v_{a,i} : 1 \leq i \leq Q\}$ induces a complete graph K_Q, which is drawn using Corollary 7 (with the dimensions permuted) in the box

$$[2a, 2a + P] \times [aQ + 1, (a + 1)Q] \times [0, -c\,Q^2/P]\ ,$$

for some constant c. For all $1 \leq a < b \leq P$, orient each edge $e = (v_{a,i}, v_{b,j})$, and position the bend for e at

$$r_e\ =\ (2a + 1,\ bQ + j,\ 4^{P-a}\,Q - i)\ ,$$

as illustrated in Figure 2. We say $v_{a,i}r_e$ is an *outgoing* segment at $v_{a,i}$, and $r_e v_{b,j}$ is an *incoming* segment at $v_{b,j}$.

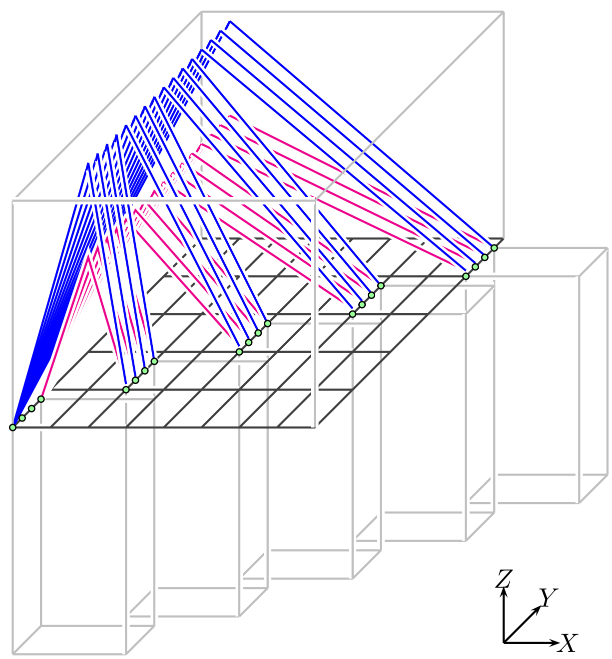

Figure 2: Construction of 1-bend drawing of K_n.

Thus the bounding box is $\mathcal{O}(P) \times \mathcal{O}(n) \times \mathcal{O}(4^P Q + Q^2/P)$, which is $\mathcal{O}(\log n) \times \mathcal{O}(n) \times \mathcal{O}(n^{3/2}/\log n + n^2/\log^3 n)$, which is $\mathcal{O}(\log n) \times \mathcal{O}(n) \times \mathcal{O}(n^2/\log^3 n)$. Hence the volume is $\mathcal{O}(n^3/\log^2 n)$. It remains to prove that there are no edge crossings. By Corollary 7 all edges below the $Z = 0$ plane do not cross. We now only consider edges above the $Z = 0$ plane.

Each point in an outgoing segment at $v_{a,i}$ has an X-coordinate in $[2a, 2a + 1]$. Thus an outgoing segment at some vertex v_{a_1,i_1} does not intersect an outgoing segment at some vertex v_{a_2,i_2} whenever $a_1 \neq a_2$. Clearly an outgoing segment at v_{a,i_1} is not coplanar with an outgoing segment at v_{a,i_2} whenever $i_1 \neq i_2$, and thus these segments do not cross. Since each bend is assigned a unique gridpoint, any two outgoing segments at the same vertex $v_{a,i}$ do not cross. Thus no two outgoing segments cross.

Each point in an incoming segment at $v_{b,j}$ has a Y-coordinate of $bQ + j$. Thus incoming segments at distinct vertices do not cross. Since each bend is assigned a unique gridpoint, any two incoming segments at the same vertex do not cross. Thus no two incoming segments cross.

To prove that an incoming segment does not cross an outgoing segment, we claim that in the projection of the edges on the $Y = 0$ plane, an incoming segment does not cross an outgoing segment. In the remainder of the proof we work solely in the $Y = 0$ plane, and use (X, Z) coordinates.

The projection in the $Y = 0$ plane of an outgoing segment at a vertex $v_{a,i}$ is the segment

$$s_1 = (2a, 0) \to (2a + 1, 4^{P-a}\,Q - i) \ .$$

The projection in the $Y = 0$ plane of the incoming segment of an edge $(v_{c,k}, v_{d,\ell})$ is the segment

$$s_2 = (2c + 1, 4^{P-c}\,Q - k) \to (2d, 0).$$

For there to be a crossing clearly we must have $c < a < d$. To prove that there is no crossing it suffices to show that the Z-coordinate of s_2 is greater than the Z-coordinate of s_1 when $X = 2a + 1$. Now s_2 is contained in the line

$$Z = \frac{4^{P-c}\,Q - k}{2c + 1 - 2d}(X - 2d) \ .$$

Thus the Z-coordinate of s_2 at $X = 2a + 1$ is at least

$$\frac{4^{P-c}\,Q - Q}{2c + 1 - 2d}(2a + 1 - 2d) \ .$$

Thus it suffices to prove that

$$\frac{4^{P-c}\,Q - Q}{2c + 1 - 2d}(2a + 1 - 2d) \ > \ 4^{P-a}\,Q \ . \qquad (1)$$

Clearly (1) is implied if it is proved with $a = c + 1$ and $d = c + 2$. In this case, (1) reduces to

$$\frac{4^{P-c} - 1}{3} \ > \ 4^{P-c-1} \ .$$

That is, $4^{P-c-1} > 1$, which is true, since $c \leq P - 2$. This completes the proof.

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Banana Spiders: A Study of Connectivity in 3D Combinatorial Rigidity

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Abstract

Finding a combinatorial test for rigidity in 3D is an open problem. We prove that vertex connectivity cannot be used to construct such a test by describing a class of mechanisms that increase the vertex connectivity of flexible graphs to 5. Our result is tight, as minimally rigid graphs in 3D can be at most 5-connected.

1 Introduction

In two dimensions, combinatorial rigidity is well understood: Laman's condition on the number and distribution of edges is both necessary and sufficient for determining if a framework is rigid. In three dimensions, however, finding a test for combinatorial rigidity has proved elusive. Little has been published on the failed attempts. In this paper we show that vertex connectivity does not help us in our goal: 3-connectivity together with the 3D extension to Laman's condition is insufficient, and 4- and 5-connectivity are neither sufficient nor necessary; a minimally rigid graph cannot be greater than 5-connected.

There are many models of rigidity. We examine first-order rigidity of bar-joint frameworks [3, 5]. Mathematically, a framework is defined as graph with an embedding in $\Re^d$. Once embedded, the edges of the graph become fixed length bars connected at flexible joints. Knowing whether a framework is flexible or rigid, i.e. whether or not there exists an edge-length preserving deformation that changes the distances between some non-adjacent vertices, is useful in many applications, such as designing bridges and other structures. If a graph G has a rigid embedding, then almost all embeddings of G produces a rigid framework. Thus we would like to assume a *generic embedding* (see [3, 5]), and determine whether or not a framework is rigid based solely on the graph of vertices and edges. (We call a graph *rigid* in $\Re^d$ if there exists an embedding in $\Re^d$ that gives a rigid framework.)

In 1970, Laman published a condition that can be used to test whether a graph is rigid in $\Re^2$:

Condition 1 (Laman, [3, 4]) *A graph $G = (V, E)$ is rigid for dimension 2 if and only if there is a subset E' of E such that:*

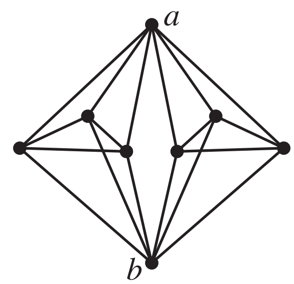

Figure 1: The double banana, with an implied hinge edge through a and b.

1. $|E'| = 2|V| - 3$, *and*
2. *for all* $E'' \subset E'$ *where* $|V(E'')| \geq 2$*, we have* $|E''| \leq 2|V(E'')| - 3$.

This condition, known as Laman's condition, is both necessary and sufficient. Note that the graph $G' = (V, E')$ is *minimally rigid*: removing any edge from G' gives a flexible graph. Embedded generically, a minimally rigid graph produces an isostatic framework [5].

Modifying Laman's condition for 3D, we get:

Condition 2 ([3]) *A graph $G = (V, E)$ is rigid for dimension 3 if and only if there is a subset E' of E such that:*

1. $|E'| = 3|V| - 6$, *and*
2. *for all* $E'' \subset E'$ *where* $|V(E'')| \geq 3$*, we have* $|E''| \leq 3|V(E'')| - 6$.

We refer to Condition 2 as Laman's condition, and call graphs satisfying this condition *Laman graphs*. Although Laman's condition is necessary, it is no longer sufficient. The *double banana* [2], shown in Figure 1, is the classic example of a framework that satisfies Laman's condition, yet is flexible.

The double banana is the smallest example where Laman's condition is insufficient, but what are others? Lacking a necessary and sufficient extension of Laman's condition to 3D, we would at least like to characterize the cases where Laman's condition is not sufficient.

A natural question is whether triangles are required for rigidity. Euler's formula shows that planar graphs require at least one triangle to be rigid in 2D, and must be fully triangulated to be rigid in 3D. The bipartite graph $K_{3,3}$, however, was known in the 19th century to be infinitesimally rigid in 2D. Bolker and Roth [1] proved that triangles are also not

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