On the Fréchet distance of a set of curves

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Abstract

The *Fréchet distance* of two curves measures the resemblance of the curves and is known to have applications in shape comparison and recognition. We extend this notion to a set of curves and show how it can be computed and approximated.

1 Introduction

A *curve* is a continuous mapping $f\colon [a,b] \to \mathbb{R}^d$ with $a, b \in \mathbb{R}$ and $a < b$. As a measure for the resemblance of curves, Alt and Godau have considered the so-called Fréchet distance δ_F.

Definition 1 *Let $f\colon [a,a'] \to \mathbb{R}^d$ and $g\colon [b,b'] \to \mathbb{R}^d$ be two curves. The* Fréchet distance*, denoted $\delta_F(f,g)$, is defined as*

$$\delta_F(f,g) := \inf_{\substack{\alpha\colon [0,1]\to[a,a'] \\ \beta\colon [0,1]\to[b,b']}} \max_{t\in[0,1]} \|f(\alpha(t)) - g(\beta(t))\|,$$

where α, β range over continuous and increasing functions with $\alpha(0) = a$, $\alpha(1) = a'$, $\beta(0) = a$ and $\beta(1) = b'$. The functions α, β are also called parametrization *functions.*

For polygonal curves P and Q, consisting of p and q edges, Alt and Godau [2] developed an algorithm which computes their Fréchet distance in time $O(pq \log(pq))$. We will review this algorithm in Section 3.

Another possibility of measuring the resemblance of two curves is the via the so-called *Hausdorff distance* δ_H between the set of points which make up the two curves. For two bounded sets $A, B \subset \mathbb{R}^2$,

$$\delta_H(A,B) = \max\left(\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right),$$

The Hausdorff distance between two polygonal curves P and Q, consisting of p and q edges, can be computed in time $O(p+q)\log(p+q)$ [1]. The Hausdorff distance considers the curves just as point sets.

While in some applications the Hausdorff distance is a suitable measure of similarity, there are others in which is not: especially when the course of the curves is important — such as when they are input by a digitizer — and the order of the points on each curve is relevant. The Fréchet distance, defined by Fréchet [4, 5], is more suitable in this case.

A popular and highly intuitive illustration [2] of the Fréchet distance is the following. Suppose a man is walking his dog, he is walking on one curve and the dog on the other curve. Both are allowed to adjust their speed but are not allowed to go backwards. Then the Fréchet distance of the two curves is the minimum length of a leash that is necessary.

Since δ_F is symmetric and the triangle inequality holds [5], δ_F is a metric on the set of all curves if two curves which differ only in their parametrization are regarded as equal [2].

The following definition is a natural extension of the Fréchet distance to a set of $m \geq 3$ curves.

Definition 2 *For a set of m curves $\mathcal{F} = \{f_1, \ldots, f_m\}$, $f_i\colon [a_i, a_i'] \to \mathbb{R}^d$, we define their* Fréchet distance *as*

$$\delta_F(\mathcal{F}) := \inf_{\substack{\alpha_1\colon [0,1]\to[a_1,a_1'] \\ \vdots \\ \alpha_m\colon [0,1]\to[a_m,a_m']}} \max_{\substack{t\in[0,1] \\ 1\le i,j\le m}} \|f_i(\alpha_i(t)) - f_j(\alpha_j(t))\|,$$

where $\alpha_1, \ldots, \alpha_m$ range over continuous and increasing functions with $\alpha_i(0) = a_i$ and $\alpha_i(1) = a_i'$, $i = 1, \ldots, m$.

The corresponding intuitive illustration is as follows. Suppose that m points are moving, one on each of m given curves. The speed of each point may vary but no point is allowed to move backwards. Assume that all pairs of points are connected by strings of the same length. Then the Fréchet distance of the set of curves is the minimum length of a connecting string that is necessary.

Theorem 1 *Consider a finite set of $m \geq 3$ curves in arbitrary dimensions. Let $d_{ij} := \delta_F(f_i, f_j)$ and $d_{\mathcal{F}} := \delta_F(\mathcal{F})$. Then*

$$d_{\mathcal{F}} \leq \min_{1\le i\le m} \max_{1\le j<k\le m} (d_{ij} + d_{ik}).$$

This inequality is best possible: already for $m = 3$, and for any choice of numbers d_{12}, d_{13}, d_{23} satisfying the triangle inequality, there are 3 curves for which equality holds.

Section 2 contains the proof of Theorem 1, while Section 3 discusses the computation and approximation of the Fréchet distance of a set of curves.

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2 Proof of Theorem 1

The first part is immediate and follows from the fact that the complete graph K_m includes the star $K_{1,m-1}$. Select $i \in [m]$ which minimizes $\max_{1 \le j < k \le m}(d_{ij} + d_{ik})$, and let $\alpha_1, \ldots, \alpha_m$ be corresponding parametrization functions so that for any $j \in [m] \setminus \{i\}$, $d_{ij} = \max_{t \in [0,1]} \|f_i(\alpha_i(t)) - f_j(\alpha_j(t))\|$. By the triangle inequality, for any $j, k \in [m]$, $\max_{t \in [0,1]} \|f_j(\alpha_j(t)) - f_k(\alpha_k(t))\| \le d_{ij} + d_{ik}$. This implies that

$$d_{\mathcal{F}} \le \min_{1 \le i \le m} \max_{1 \le j < k \le m} (d_{ij} + d_{ik}). \tag{2}$$

Intuitively, i is chosen as the "leading" curve, or the "man" curve, while the others are the "dog" curves. Then the distance between any two dogs or between the man and any dog throughout the walk is bounded by the sum of the lengths of the longest two leashes the man holds.

We will now construct an example in which the bound is tight. Even if our bounds hold in any dimensions, the curves in this example can be selected of the simplest form, i.e., polygonal curves on the real line. For illustration however, the different segments are drawn stacked on each other, see Figure 1.

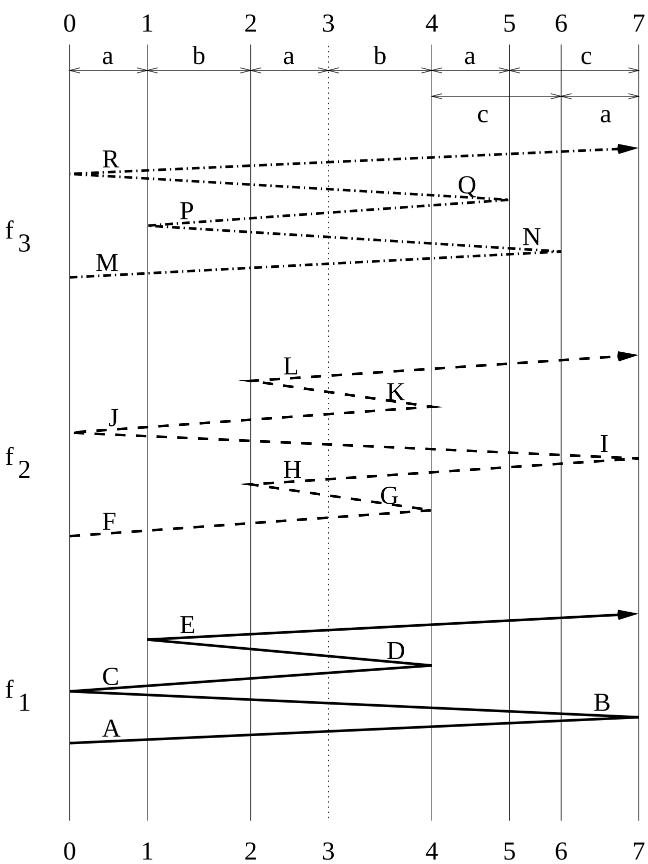

Figure 1: A set of three polygonal curves used in the proof of Theorem 1. Parameters a, b, c satisfy $a \le b \le c$ and $c \le a + b$.

The following observation is useful in obtaining lower bounds:

Observation 1 *Consider a piece f' of a curve which consists of a horizontal segment of length x, traversed from right to left. Suppose that this piece is matched to a piece of another curve which moves monotonically to the right (or which is just a single point). Then the maximum distance in this joint parametrization is at least $x/2$.*

Lemma 2 *Let $\mathcal{F} = \{f_1, f_2, f_3\}$ be the set of three polygonal curves shown in Figure 1, where $a \le b \le c$ and $c \le a + b$. Then $d_{12} = b$, $d_{13} = a$, $d_{23} = c$ and $d_{\mathcal{F}} = a + b$.*

Proof. Denote the five segments which make up f_1 by A–E. Similarly we denote by F–L and M–R the seven (resp. five) segments of f_2 and f_3.

The idea of the example is as follows: To achieve the minimum distance $d_{23} = c$, the small wiggle FGH on f_2 must be matched to the large zigzag MNP on f_3. On the other hand, the minimum distance $d_{12} = b$ can only be achieved if the small wiggle FGH on f_2 matches the straight movement A on f_1. For achieving the minimum distance $d_{13} = a$, A must be matched to M, however. It follows that not all three pairwise minimum distances can be achieved simultaneously in a joint reparametrization for all three curves.

A graphical representation of the situation is shown in Figure 2 as a three-dimensional box, representing the joint parameter space $(\alpha_1, \alpha_2, \alpha_3)$ of three points moving on the three curves. The three sides of the box are *free-space diagrams* [2] for pairs of curves with a threshold value $\varepsilon = a+b$. White regions (the free space) correspond to pairs of points $f_i(\alpha_i)$ and $f_j(\alpha_j)$ with distance at most ε, and shaded areas are forbidden areas where the distance is too big. A solution with Fréchet distance at most ε is represented by a path from the origin (in the center of the picture) to the opposite corner of the box which is monotone in each direction, and for which the projection to each coordinate hyperplane lies in the free space. The projections of two such paths is shown in the figure. (They have the same projection on the α_1–α_2 plane.) One sees that certain passages are about to become blocked if ε is decreased below $a+b$, for example, point 1 in the α_2–α_3 plane. The segment labeled 2 in the α_2–α_3 plane and in the α_1–α_3 plane will be unpassable because it is no longer monotone when ε is smaller than $a + b$. Some other segments of this kind are shown as dotted lines. It is apparent that for $\varepsilon < a + b$ the only remaining monotone paths in the α_1–α_3 plane go around the obstacles like the path through point 1 shown in these pictures. One can then check that this is inconsistent with a monotone path through space whose projection on the α_1–α_2 plane and on the α_2–α_3 plane lies in the free space.

In the following we give a detailed and elementary proof that does not make recourse to the free-space diagram. We denote different points on the curves by their labels $\in \{0, 1, 2, 3, 4, 5, 6, 7\}$ indexed by the segment to which they

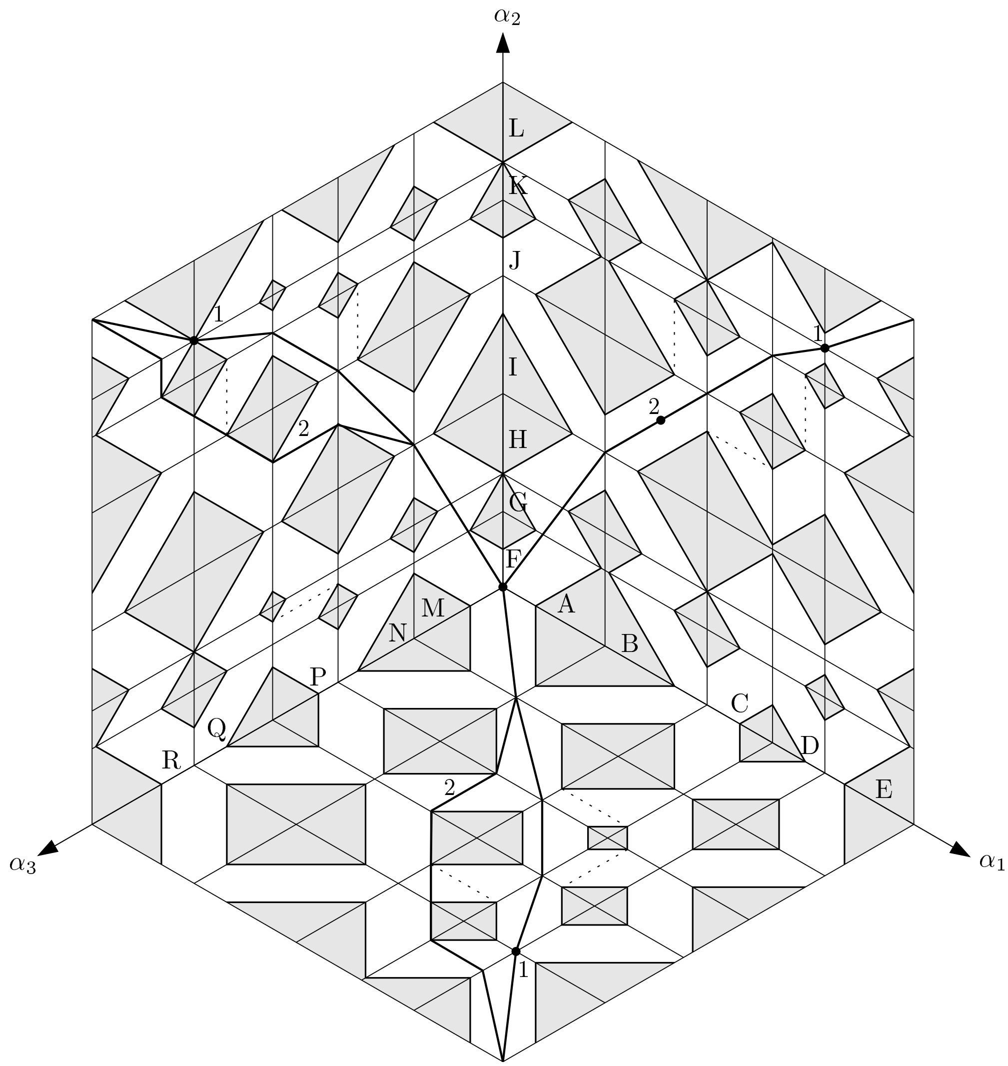

Figure 2: The pairwise free-space diagrams for the curves f_1, f_2, and f_3, for parameter values $a = 1$, $b = 1.5$, $c = 1.8$ and threshold value $\varepsilon = a + b$.

belong, e.g., $7_A \equiv 7_B$ denotes the common endpoint of segments A and B on curve f_1, at location 7.

It is sufficient to prove $d_{12} \leq b$, $d_{13} \leq a$, $d_{23} = c$, and $d_{\mathcal{F}} \geq a+b$. Equality for d_{12}, d_{13}, and $d_{\mathcal{F}}$ follows then from the relation $d_{\mathcal{F}} \leq d_{12} + d_{13}$ (since then, $a + b \leq d_{\mathcal{F}} \leq d_{12} + d_{13} \leq a + b$).

We start by proving that $d_{12} \leq b$, by exhibiting a schedule S_{12} for a pair of moving points $p \in f_1$, $q \in f_2$ such that at any time the distance between them is at most b. In some parts of this schedule the two points move at the same speed along a common portion of the two curves, in others only one point is moving while the other stays put. S_{12}: $p_1, p_2 \colon 0, 1, 2, 3$; $p_2 \colon 3, 4, 3, 2, 3$; $p_1, p_2 \colon 3, 4, 5, 6, 7, 6, 5, 4, 3, 2, 1, 0, 1, 2, 3, 4, 3, 2$; $p_1 \colon 2, 1, 2$; $p_1, p_2 \colon 2, 3, 4, 5, 6, 7$. The maximum distance between the two points in this schedule is b, as claimed.

The proofs of $d_{13} \leq a$ and of $d_{23} \leq c$ follow a similar pattern. The following schedule S_{13} proves that $d_{13} \leq a$. S_{13}: $p_1, p_3 \colon 0, 1, 2, 3, 4, 5, 6$; $p_1 \colon 6, 7, 6$; $p_1, p_3 \colon 6, 5, 4, 3, 2, 1$; $p_1 \colon 1, 0, 1$; $p_1, p_3 \colon 1, 2, 3, 4$; $p_3 \colon 4, 5, 4$; $p_1, p_3 \colon 4, 3, 2, 1$; $p_3 \colon 1, 0, 1$; $p_1, p_3 \colon 1, 2, 3, 4, 5, 6, 7$. It is clear that the maximum distance between the two points in this schedule is a.

The following schedule S_{23} proves that $d_{23} \leq c$. S_{23}: $p_2, p_3 \colon 0, 1, 2, 3, 4$; $p_3 \colon 4, 5, 6, 5, 4$; $p_2, p_3 \colon 4, 3, 2$; $p_3 \colon 2, 1, 2$; $p_2, p_3 \colon 2, 3, 4, 5$; $p_2 \colon 5, 6, 7, 6, 5$; $p_2, p_3 \colon 5, 4, 3, 2, 1, 0, 1, 2, 3$; $p_2 \colon 3, 4, 3, 2, 3$; $p_2, p_3 \colon 3, 4, 5, 6, 7$. The maximum distance between the two points in this schedule is $\max\{a, b, c\} = c$.

We now show that $d_{23} \geq c$. Consider any schedule for p_2, p_3, and let t_1 be the earliest time when p_3 is in 1_P, t_2 be the earliest time when p_3 is in 5_Q and t_3 be the earliest time when p_3 is in 0_R. We analyze the position of p_2 at t_1. If $p_2 \in F$ at t_1, since p_3 has reached 6_N at some $t < t_1$, $|p_2 p_3| \geq c$ at t. If $p_2 \in G \cup H$ at t_1, then p_2 must reach 7_I at some $t \in [t_1, t_2]$, otherwise $|p_2 p_3| \geq a + b \geq c$ after t_2; hence at t, $|p_2 p_3| \geq |5_Q 7_I| = c$. If $p_2 \in I \cup J$ at

t_1, then at t_2, $p_2 \in [3_J 4_J] \cup [4_K 3_K]$; it follows that at t_3, $|p_2p_3| \geq |0_R 2_L| = a + b \geq c$. If $p_2 \in K \cup L$ at t_1, we also have at t_3, $|p_2p_3| \geq |0_R 2_L| = a + b \geq c$. We conclude that $d_{23} \geq c$, thus $d_{23} = c$.

Finally we have to prove that $d_{\mathcal{F}} \geq a + b$. Some of the arguments are the same as for proving $d_{23} \geq c$. Consider any schedule for p_1, p_2, p_3, and let t_1, t_2, t_3 be as above. We analyze the position of p_2 at t_1.

Assume that $p_2 \in F$ at t_1. Then p_3 is in 6_N at some $t < t_1$. If $p_2 \in [0_F 3_F]$ at t, then $|p_2p_3| \geq b + c \geq a + b$ at t. If $p_2 \in [3_F 4_F]$ at t, then $|p_2p_3| \geq a + b$ at t_1.

Assume that $p_2 \in G \cup H$ at t_1. More precisely, $p_2 \in [3_G 2_G] \cup [2_H 3_H]$ at t_1. We now also analyze the position of p_1 prior to t_1. Note that p_2 is in 4_G at some $t < t_1$. If p_1 is on A at t_1, we distinguish four cases. If $p_1 \in [0_A 2_A]$ at t_1, we have $|p_1p_2| \geq a + b$ at t. If $p_1 \in [2_A 3_A]$ at t_1, we have $|p_1p_3| \geq b + c \geq a + b$ at some $t' < t_1$, when p_3 is in 6_N. If $p_1 \in [3_A 7_A]$ at t_1, we have $|p_1p_3| \geq a + b$ at t_1. If p_1 is on $B \cup C \cup D \cup E$ at t_1, we have $|p_1p_2| \geq a + c \geq a + b$ at some $t'' < t_1$, when p_1 is in 7_A.

The remaining case is $p_2 \in I \cup J \cup K \cup L$ at t_1. If $p_2 \in I$ at t_2, then $|p_2p_3| \geq 2(a+b)/2$ by Observation 1. If $p_2 \in J \cup K$ at t_2, we must have $p_2 \in [3_J 4_J] \cup [4_K 3_K]$ at t_2; it follows that at t_3, $|p_2p_3| \geq |0_R 2_L| = a + b$.

We conclude that $d_{\mathcal{F}} \geq a + b$, thus $d_{\mathcal{F}} = a + b$. □

3 Approximating the Fréchet Distance

Consider a set of m polygonal curves $\mathcal{F} = \{f_1, \ldots, f_m\}$, $f_i \colon [a_i, a_i'] \to \mathbb{R}^d$, where n_i is the number of segments of f_i, $i = 1, \ldots, m$. As mentioned in the introduction [2] presents an algorithm for computing the Fréchet distance between two polygonal curves with p and q segments in time $O(pq \log(pq))$. Their algorithm makes use of one for solving the easier *decision problem*, namely: given polygonal curves P and Q and some $\epsilon \geq 0$, decide whether $\delta_F(P, Q) \leq \epsilon$. This can be answered in time $O(pq)$. The algorithm for actually computing $\delta_F(P, Q)$ for given polygonal curves P and Q makes use of the decision algorithm and the technique of *parametric search* of Megiddo [6], accompanied by a speedup technique due to Cole [3]. The resulting final algorithm has time complexity $O(pq \log(pq))$.

Returning now to the decision problem, $\delta_F(P, Q) \leq \epsilon$ if and only if there exists a curve from $(0, 0)$ to (p, q) in the free-space diagram in the plane which is monotone in both coordinates.

To compute the Fréchet distance of a set of curves, this approach can be adapted. In answering the decision problem, one would have to check whether there exists a curve from $(0, \ldots, 0)$ to $(n_1, \ldots, n_m)$ in the free-space diagram in $\mathbb{R}^m$ which is monotone in all m coordinates. This takes $O(n_1 \ldots n_m)$ time, and the resulting final algorithm has time complexity $O(n_1 \ldots n_m \log(n_1 \ldots n_m))$. Note that this is prohibitive even for small values of m, e.g. for $m = 10$, and even for the decision problem.

Therefore we outline the following simple algorithm which computes the Fréchet distance of a set of curves approximately, based on our bound in Theorem 1. The algorithm computes all pairwise Fréchet distances and outputs $\min_{1 \leq i \leq m} \max_{1 \leq j < k \leq m} (d_{ij} + d_{ik})$. Since $d_{\mathcal{F}} \geq \max_{1 \leq i < j \leq m} d_{ij}$, the approximation ratio is 2. The running time of the approximation algorithm is $O(\sum_{1 \leq i < j \leq m} n_i n_j \log(n_i n_j))$, i.e., much better than that of the exact algorithm previously mentioned.

4 Extensions and open questions

Our Definition 2 of the Fréchet distance of a set of curves measures the maximum *diameter* of the point set $\{ f_i(\alpha_i(t)) \mid i = 1, \ldots, m \}$ at all times t. One can ask similar questions for other "size" measures of a point set, for example the radius of the smallest enclosing ball, or the sum of all pairwise distances.

We have shown that using pairwise Fréchet distances yields a factor-2 approximation for the Fréchet distance of a set of curves (which follows from the triangle inequality). Is it possible to get a better approximation factor by, for example, considering all triple-wise Fréchet distances d_{ijk}? What is the best bound for d_{1234} in terms of d_{123}, d_{124}, d_{134}, and d_{234}?

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