An Anisotropic Cardinality Bound for Triangulations

Steven E. Pav*

Abstract

A bound on the cardinality of planar triangulations with minimum angle α is considered. It is shown that a certain integral over the edges of the triangulation is bounded by $\mathcal{O}\left((1/\alpha)\log(1/\alpha)\right)$ times the number of vertices. This is an alteration of a result by Mitchell [6], and allows an optimality bound on triangulations which are "well-graded" to the local feature size of an input domain [7]. In particular, this result allows some improvement on the output guarantees of a robust version of Ruppert's Algorithm [8, 7]. A pathological triangulation proves that the bound is tight.

1 Introduction

The problem of *2D guaranteed quality meshing* is motivated by the finite element method, which requires discretization of a continuous domain into triangles. Success of this method depends on the "quality" of the mesh, which is determined by the minimum mesh angle [1]. The quality meshing problem can be summarized as follows: given input, $\mathcal{N} = (\mathcal{P}, \mathcal{S})$ consisting of a set of points, $\mathcal{P}$, and set of segments, $\mathcal{S}$, which intersect at endpoints only, generate a triangulation $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, such that $\mathcal{P} \subseteq \mathcal{V}$, segments of $\mathcal{S}$ are represented as the union of edges of $\mathcal{E}$, there are no (or few) small angles in $\mathcal{T}$, and the cardinality of $\mathcal{V}$ is reasonably small.

Mesh generation often proceeds as follows: to the point set $\mathcal{P}$, add Steiner Points in a systematic way then return the Delaunay Triangulation of the resultant point set [4, 8, 9, 2, 3, 7]. Many of these approaches have weak cardinality guarantees. For others, only convergence is established; cardinality of the output point set has not been bounded.

Mitchell improved the cardinality bound for output of Ruppert's Algorithm [6, 8]. Throughout this notice, let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a planar triangulation with minimum angle α. Define local feature size with respect to $\mathcal{T}$, $\mathrm{lfs}_{\mathcal{T}}(z)$, as the radius of the smallest circle centered at z which intersects two *disjoint* features of $\mathcal{T}$. (See Figure 1.) Mitchell showed that

$$\int_{\Omega} \frac{1}{\mathrm{lfs}_{\mathcal{T}}^2(z)} \mathrm{d}z = \mathcal{O}\left(\frac{1}{\alpha}\right) |\mathcal{V}|, \qquad (1)$$

where Ω is the region of the plane covered by the triangulation $\mathcal{T}$.

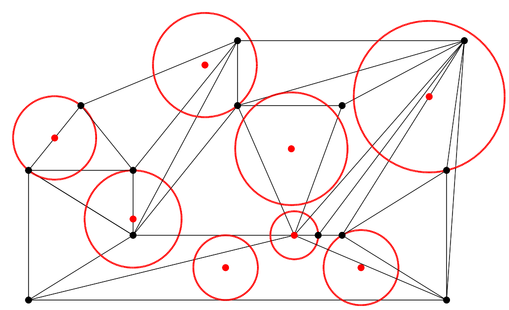


Figure 1: For a triangulation, about a number of chosen points in the plane, a circle is shown whose radius is local feature size of the center point. For clarity of illustration points are chosen so circles do not intersect.

By considering ball packing around Steiner Points, it was shown that the left hand side of equation 1 bounds the output cardinality of Ruppert's Algorithm, up to a (very large) constant [6, 8]. Better still, this result establishes an "optimality guarantee" for Ruppert's Algorithm: if another algorithm outputs a mesh respecting the input domain and with minimum angle α, then Ruppert's Algorithm can only be worse by a factor of $\mathcal{O}(1/\alpha)$ [6, 8].

The optimality bounds degenerate if Ruppert's Algorithm is generalized to accept input with nonobtuse angles [7, 5]. Consider Figure 2; If (a, b) and (b, c) are input segments subtending a small angle, then the Steiner Points x and y can be very close. An isotropic ball packing will overestimate the output cardinality, moreso if $\angle abc$ is very small. However, the distance from x to any other Steiner Points along its supporting input segment, (a, b), may be much larger, as shown.

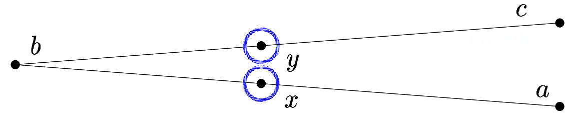


Figure 2: Ball packing overestimates point cardinality.

To improve the cardinality bound, it was shown that

$$\sum_{s \in \mathcal{S}} \int_{s} \mathrm{lfs}_{\mathcal{N}}^{-1}(z)\, \mathrm{d}z \qquad (2)$$

is an upper bound on the number of Steiner Points added

*Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112, USA. (spav@ucsd.edu) Partial funding for this work was provided by National Science Foundation Grants DMS–0208586 and CCR–9902091.

by Ruppert's Algorithm to segments of $\mathcal{S}$, up to a constant [7]. In this notice we demonstrate the following bound, for triangulation $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ with minimum angle α:

$$\int_E \frac{1}{\text{lfs}_{\mathcal{T}}(z)} \mathrm{d}z = \mathcal{O}\left(\frac{1}{\alpha} \log\left(\frac{1}{\alpha}\right)\right) |\mathcal{V}|, \qquad (3)$$

where $E = \cup_{e \in \mathcal{E}} e$. If $\mathcal{T}$ is a mesh solving the quality meshing problem for an input $\mathcal{N}$, then the left hand side of equation 3 is an upper bound for quantity (2), because each segment of $\mathcal{N}$ is the union of edges of $\mathcal{T}$, and the local feature size with respect to $\mathcal{T}$ is smaller than that with respect to $\mathcal{N}$.

We also present a lower bound to show that equation 3 cannot be improved asymptotically. In the interest of brevity, we present an abbreviated analysis; the full version may be found elsewhere [7].

2 The Whirl

Throughout this notice assume that $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is a triangulation with minimum angle α. Moreover, assume that the triangulation is maximal, *i.e.*, that no edge may be added to $\mathcal{E}$ without crossing an existing edge.[1] The following lemma is due to Mitchell.

Lemma 1 (Mitchell [6]) *Let $e, f \in \mathcal{E}$ share a common endpoint in a triangulation with minimum angle α. Then*

$$\frac{|e|}{|f|} \leq (2\cos\alpha)^{\frac{\angle ef}{\alpha}}.$$

The bound of Mitchell's lemma involves a parametric shape of some interest:

Definition 2 (α-whirl) *For p, q and minimum angle α, define the α-*whirl *centered at p to be the set*

$$\mathcal{W}(p,q) = \left\{ x \;\middle|\; |p-x| = |p-q|\,(2\cos\alpha)^{\frac{-\angle qpx}{\alpha}} \right\},$$

where it is assumed $0 \leq \angle qpx \leq \pi$. Thus the α-whirl can be imagined as a parametric curve defined in polar coordinates by

$$(\theta, |p-q|\, g_\alpha(|\theta|)),$$

where p is the origin of the coordinate system, q is on the positive x-axis, $\theta \in (-\pi, \pi]$, $g_\alpha(\theta) = \exp(-h_\alpha \theta)$, where $h_\alpha = (\ln 2\cos\alpha)/\alpha$. An α-whirl is shown in Figure 3.

The α-whirl is two pieces of an equiangular spiral of angle $\arctan(1/h_\alpha)$, *thus if z is a point on the curve, then segment (p, z) forms angle* $\arctan(1/h_\alpha)$ *with the tangent to the α-whirl at z.*

Now we consider the question: what is the distance from an α-whirl to a point inside of it? The answer will be used in establishing upper and lower bounds on the integral $\int_E \text{lfs}_{\mathcal{T}}^{-1}(z)\, \mathrm{d}z$.

[1] Mitchell drops this requirement when the input domain is a polygon with polygonal holes. In this case the holes represent regions where the output mesh is not maximal.

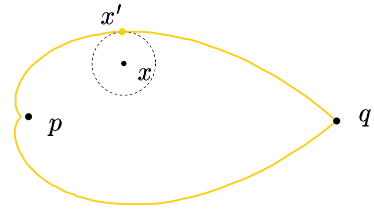


Figure 3: The α-whirl, $\mathcal{W}(p,q)$, with $\alpha \approx 27.6°$ is shown. For point x, consider the nearest point on the α-whirl, here marked x'. We will find x' in terms of $\angle qpx$ and distance $|p-x|$. Either (x, x') is normal to the α-whirl at x', or x' is the point collinear to p, q, at the "pinch point" of the α-whirl.

Consider the α-whirl, $\mathcal{W}(p,q)$. Adjust the units so that $|p-q| = 1$. Let z be the point $(\phi, \lambda g_\alpha(\phi))$, for $\phi \in [0,\pi]$, $\lambda \in [0,1]$. For this choice of ϕ, λ, we only need to consider the distance to the "upper" half of the α-whirl, so we can restrict θ to be in $[0,\pi]$. The squared distance from z to point $(\theta, g_\alpha(\theta))$ is

$$f_{\lambda,\phi}(\theta) \quad =_{\text{df}} \quad |(\phi, \lambda g_\alpha(\phi)) - (\theta, g_\alpha(\theta))|^2 \qquad (4)$$

To minimize $f_{\lambda,\phi}(\theta)$ we look for zeroes of its derivative. Letting $\psi = \theta - \phi$, we set the derivative to zero and find

$$\lambda = j_\alpha(\psi) =_{\text{df}} \frac{h_\alpha g_\alpha(\psi)}{\sin\psi + h_\alpha \cos\psi}.$$

Tedious analysis yields

Claim 3 *Let θ minimize $f_{\lambda,\phi}(\theta)$, for $\phi \leq \pi/2$. Then*
- *if $\lambda \leq j_\alpha(\pi/2)$, then $\theta = \pi$,*
- *if $j_\alpha(\pi/2) < \lambda \leq j_\alpha(\arctan h_\alpha)$, then θ is either π or $\phi + \psi_1$, and*
- *if $j_\alpha(\arctan h_\alpha) < \lambda \leq 1$, then $\theta = \phi + \psi_1$,*

where ψ_1 is the root of $\lambda = j_\alpha(\psi)$ which is less than $\pi/2$.

3 An Upper Bound

In this section, the α-whirl is used to yield an upper bound for the integral $\int_E \text{lfs}_{\mathcal{T}}^{-1}(z)\, \mathrm{d}z$. First it is shown that if x is a point on an edge, (p,q), of the triangulation, with $|x-p| \leq |x-q|$, then there is point q' on the segment such that the distance from x to the α-whirl $\mathcal{W}(p,q')$ is a *lower* bound on $\text{lfs}(x)$. This is done simply in Lemma 4 by showing that no edge or point of the mesh disjoint from (p,q) or the point p is inside this α-whirl. Then in Theorem 5, the distance from a point to a α-whirl is used to bound the integral.

Lemma 4 *Let $e = (p,q)$ be an edge of a triangulation with minimum angle α. Let q' be the point on (p,q) such that $|p-q'| = \frac{\sqrt{3}}{2}|p-q|$. Then there is no edge or vertex of the triangulation which is disjoint from e or from p and contained inside the closed curve $\mathcal{W}(p,q')$.*

Proof. Let $\{q_i\}_{i=0}^{n-1}$ be the set of vertices of the triangulation such that (p, q_i) is an edge of the triangulation; moreover, assume $q_0 = q$, and the vertices q_i are ordered counterclockwise around p. All the triangles in the triangulation which have p as a corner are of the form Δpq_iq_{i+1} for some $0 \le i < n$, where q_n is read to be q_0.

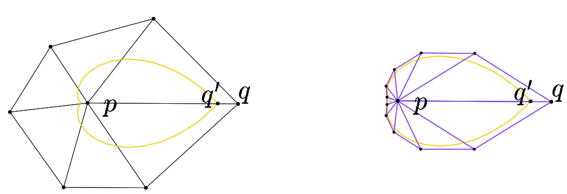
It suffices to show that no point of (q_i, q_{i+1}) is inside $\mathcal{W}(p, q')$. Let θ be the counterclockwise angle $\angle q_0pq_i$; without loss of generality assume this is less than counterclockwise angle $\angle q_0pq_{i+1}$, and $\theta \in [0, \pi)$.

Let z be a point on (q_i, q_{i+1}). Without loss of generality assume that $\angle q_0pz \in [\theta, \pi]$. It suffices to show that $|p-z| \ge \frac{\sqrt{3}}{2}|p-q_0|\, g_\alpha(\angle q_0pz)$. Let $\phi = \angle q_ipz$. By assumption, $0 \le \phi \le \pi - 2\alpha$. Let $\psi = \angle pq_iq_{i+1}$. Using the sine rule and Lemma 1, we have

$$|p-z| \ge \frac{|p-q_0|\, g_\alpha(\theta)\sin\psi}{\sin(\phi+\psi)}.$$

Since $\angle q_0pz = \theta + \phi$, it suffices to prove that $\frac{\sin\psi}{\sin(\phi+\psi)} \ge \frac{\sqrt{3}}{2} g_\alpha(\phi)$. So we attempt to minimize $k_\psi(\phi) =_{\text{df}} \frac{\sin\psi}{\sin(\phi+\psi)} g_\alpha(-\phi)$.

Some analysis shows that this is bounded from below by $\sqrt{3}/2$. □



(a) "Normal" (b) "Pathological" PSfrag replacements

Figure 4: Lemma 4 illustrated. In (a), a "normal" triangulation, with $\mathcal{W}(p, q')$, where $|p-q'| = \frac{\sqrt{3}}{2}|p-q|$. In (b), a more pathological triangulation: all but one triangle are isosceles with base angles equal to α.

Theorem 5 *Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a triangulation with minimum angle α, where $\alpha \le \pi/6$. Let $\mathrm{lfs}_{\mathcal{T}}(z)$ be the local feature size of a point z with respect to the triangulation, and let $E = \cup_{e\in\mathcal{E}} e$. Then*

$$I =_{\text{df}} \int_E \frac{1}{\mathrm{lfs}_{\mathcal{T}}(z)} \mathrm{d}z = \mathcal{O}\left(\frac{1}{\alpha}\log\left(\frac{1}{\alpha}\right)\right)|\mathcal{E}|.$$

Proof. Let $e = (p, q)$ be an edge of the triangulation. It suffices to show that

$$\int_p^m \frac{1}{\mathrm{lfs}_{\mathcal{T}}(z)}\mathrm{d}z = \mathcal{O}(h_\alpha \ln(h_\alpha + 1/h_\alpha))$$

where m is the midpoint of the segment.

Let q' be the point on segment (p, q) such that $|p-q'| = \frac{\sqrt{3}}{2}|p-q|$. By Lemma 4, if z is a point on (p, m), then $\mathrm{lfs}_{\mathcal{T}}(z)$ is at least the distance from z to $\mathcal{W}(p, q')$. Let $\lambda = \frac{2}{\sqrt{3}}\frac{|p-z|}{|p-q|}$. Then

$$I \le \frac{\sqrt{3}\,|p-q|}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{|p-q'|\sqrt{f_{\lambda,0}(mn(\lambda))}}\mathrm{d}\lambda,$$

where $mn(\lambda)$ is the θ which minimizes $f_{\lambda,0}(\theta)$. (*cf.* equation 4.)

The integral is split, using Claim 3 to identify $mn(\lambda)$:

$$I \le \int_0^{j_\alpha(\arctan h_\alpha)} \frac{1}{\sqrt{f_{\lambda,0}(\pi)}}\mathrm{d}\lambda + \int_{j_\alpha(\pi/2)}^{\frac{1}{\sqrt{3}}} \frac{1}{\sqrt{f_{\lambda,0}(\psi_1)}}\mathrm{d}\lambda,$$

where, again, $\psi_1 < \pi/2$ is the root of $\lambda = j_\alpha(\psi)$.

Substituting the value of $f_{\lambda,0}(\pi)$ and making a change of variables gives

$$I \le \ln\left|\frac{3}{2g_\alpha(\pi)}\right| + \int_{j_\alpha^{-1}(\frac{1}{\sqrt{3}})}^{\pi/2} \frac{h_\alpha\sqrt{h_\alpha^2+1}\cos\psi_1\ \mathrm{d}\psi_1}{\sin\psi_1(\sin\psi_1 + h_\alpha\cos\psi_1)}.$$

Trigonometric substitutions then yield

$$\int_{j_\alpha^{-1}(\frac{1}{\sqrt{3}})}^{\pi/2} \frac{h_\alpha\cos\psi\ \mathrm{d}\psi}{\sin\psi(\sin\psi + h_\alpha\cos\psi)} = \mathcal{O}(\ln(h_\alpha + 1/h_\alpha)).$$

□

4 A Lower Bound

By considering a triangulation in which the local feature size is bounded from above by the distance to an α-whirl, the bound of the previous section can be shown to be optimal.

Definition 6 *Let $n \ge 4$ be an even integer, let $\alpha = \pi/n$, and let $\mathcal{V}_n = \{p\} \cup \{q_i\}_{i=0}^{n}$, where p is the coordinate origin, and q_i is, in polar coordinates, $(\alpha i, g_\alpha(\alpha i))$. Let $\mathcal{E}_n = \{(p, q_i)\}_{i=0}^{i=n} \cup \{(q_i, q_{i+1})\}_{i=0}^{n-1}$. The* spiral mesh *on n is the triangulation $\mathcal{T}_n = (\mathcal{V}_n, \mathcal{E}_n)$. It has minimum angle α. See Figure 5.*

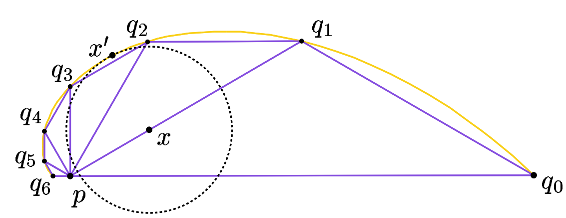


Figure 5: The spiral mesh for $n = 6$, and the top half of $\mathcal{W}(p, q_0)$, for $\alpha = \pi/6$. For point x on segment (p, q_1), $|x - x'| \ge \mathrm{lfs}_{\mathcal{T}}(x)$, where x' is the closest point to x on $\mathcal{W}(p, q_0)$. This holds when This holds when $\angle xpx' \ge \alpha$.

Given edge (p, q_i) of a spiral mesh, for many points on (p, q_i), the local feature size of the point is approximately the distance to the α-whirl, $\mathcal{W}(p, q_0)$. See Figure 5. This holds for any x such that $\angle xpx' \ge \alpha$, where x' is the point on $\mathcal{W}(p, q_0)$ closest to x. Thus the lower bound requires the same ideas as the upper bound proof.

Theorem 7 *For any $\epsilon > 0$, there is some $N \geq 4$ such that for $n \geq N$, if $\mathrm{lfs}_{\mathcal{T}}(z)$ is the local feature size with respect to the spiral mesh on n, and $0 \leq i \leq n/2$, then*

$$\int_p^{q_i} \frac{1}{\mathrm{lfs}_{\mathcal{T}}(z)} \mathrm{d}z \geq (1-\epsilon)\ln 2 \left[\frac{1}{\alpha}\ln\frac{1}{\alpha}\right],$$

where $\alpha = \pi/n$.

Proof. Assume $n \geq N \geq 4$. Let $\phi = \alpha i$. Let z be the point on (p, q_i) such that $|p - z| = \lambda g_\alpha(\phi)$. Then $\mathrm{lfs}_{\mathcal{T}}(z) \leq \sqrt{f_{\lambda,\phi}(\theta)}$ for any $\theta \in [\phi + \alpha, \pi]$. This holds because the line segment from z to the point, in polar coordinates, $(\theta, g_\alpha(\phi))$, cuts through a line segment of the form (q_j, q_{j+1}) for $i < j < n$ if $\alpha \leq \theta$. This line segment is disjoint from (p, q_i), giving the upper bound on local feature size. Then

$$I = \int_p^{q_i} \frac{1}{\mathrm{lfs}_{\mathcal{T}}(z)} \mathrm{d}z \geq g_\alpha(\phi) \int_{j_\alpha(\pi/2)}^{j_\alpha(\alpha)} \frac{1}{\sqrt{f_{\lambda,\phi}(mn(\lambda))}} \mathrm{d}\lambda,$$

where $mn(\lambda)$ is the point on the α-whirl closest to $(\phi, \lambda g_\alpha(\phi))$. The domain has been shrunk for simplicity of analysis.

Over the given range of λ, $f_{\lambda,\phi}(mn(\lambda)) \leq f_{\lambda,\phi}(\phi + \psi_1)$, where $\psi_1 \leq \pi/2$ is the root to $\lambda = j_\alpha(\psi)$. Using Claim 3, noting that $\phi \leq \pi/2$, since $i \leq n/2$, a change of variables yields

$$I \geq \sqrt{h_\alpha^2 + 1}\ \ln\tan\frac{\psi_1}{2}\Big|_\alpha^{\pi/2} - \int_\alpha^{\pi/2} \frac{\sqrt{h_\alpha^2+1}\,\mathrm{d}\psi_1}{(\sin\psi_1 + h_\alpha\cos\psi_1)}.$$

Bounding the last denominator by a linear function,

$$I \geq h_\alpha \ln\cot\frac{\alpha}{2} + \sqrt{h_\alpha^2+1}\frac{\pi/2}{1-h_\alpha}\ln\left(h_\alpha + \frac{1-h_\alpha}{\pi/2}\alpha\right)$$

For $n \geq 4$, the first term can be bounded below by $h_\alpha \ln(1/\alpha)$. Supposing that $N \geq 10$ insures $h_\alpha \geq 2$, and the second term has negative denominator. The log part can be bounded from above by $\ln(1/\alpha)$. Using $h_\alpha \geq 2$ gives

$$I \quad \geq \quad (h_\alpha - 5)\ln\frac{1}{\alpha}$$

For N sufficiently large it can be shown that $h_\alpha - 5 \geq [(1-\epsilon)\ln 2]\frac{1}{\alpha}$, establishing the theorem. □

Corollary 8 *For sufficiently small α, there exist triangulations, $(\mathcal{V}, \mathcal{E})$ with minimum angle α such that*

$$\int_E \frac{1}{\mathrm{lfs}_{\mathcal{T}}(z)}\mathrm{d}z = \Omega\left(\frac{1}{\alpha}\log\left(\frac{1}{\alpha}\right)\right)|\mathcal{V}|,$$

where $\mathrm{lfs}_{\mathcal{T}}(z)$ is the local feature size of point z with respect to the triangulation, and $E = \cup_{e\in\mathcal{E}} e$.

5 Conclusions

The bounds on the integral of $\mathrm{lfs}_{\mathcal{T}}^{-1}(z)$ can be used to show asymptotic optimality of the output of Ruppert's Algorithm. For the case where input segments meet at obtuse angles, this work only gives improvements of about an order of magnitude over the constants established by Mitchell [6, 7]. The constants, however, are too large to be of any practical use–at least 1000, and usually much larger. The usual argument is that these bounds are based on many worst-case assumptions, and the algorithm performs much better in practice. There is much room for improvement on these constants.

References

[1] I. Babuška and A. K. Aziz. On the angle condition in the finite element method. *SIAM Journal on Numerical Analysis*, 13(2):214–226, 1976.

[2] M. Bern, S. Mitchell, and J. Ruppert. Linear–size nonobtuse triangulations of polygons. In *Proc. 10th ACM Symp. Comp. Geometry*, pages 221–230. ACM Press, 1994.

[3] C. Boivin and C. F. Ollivier-Gooch. Guaranteed-quality simplicial mesh generation with cell size and grading control. *Engineering with Computers*, 17(3):269–286, 2001.

[4] L. P. Chew. Guaranteed-quality triangular meshes. Technical Report 89-983, Computer Science Department, Cornell University, 1989.

[5] G. L. Miller, S. E. Pav, and N. J. Walkington. When and why Ruppert's algorithm works. In *Proceedings of the 12th International Meshing Roundtable*, pages 91–102. Sandia National Laboratory, September 2003.

[6] S. A. Mitchell. Cardinality bounds for triangulations with bounded minimum angle. In *Sixth Canadian Conference on Computational Geometry*, pages 326–331, 1994.

[7] S. E. Pav. *Delaunay Refinement Algorithms*. PhD thesis, Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania, May 2003.

[8] J. Ruppert. A Delaunay refinement algorithm for quality 2-dimensional mesh generation. *J. Algorithms*, 18(3):548–585, 1995. Fourth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA) (Austin, TX, 1993).

[9] J. R. Shewchuk. *Delaunay Refinement Mesh Generation*. PhD thesis, School of Computer Science, Carnegie Mellon University, Pittsburgh, Pennsylvania, May 1997. Available as Technical Report CMU-CS-97-137.