Degree-Bounded Minimum Spanning Trees*

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Abstract

Given n points in the Euclidean plane, the degree-δ-MST problem asks for a spanning tree of minimum weight in which the degree of each node is at most δ. It is shown in this paper that, for any set of points in the Euclidean plane, the ratio of a degree-4-MST to a minimum spanning tree is at most $(\sqrt{2}+2)/3$.

1 Introduction

The degree-δ-MST problem is a generalization of the Hamiltonian path problem, which is NP-hard [5]. The Euclidean version of the problem in $\Re^2$ is NP-hard for $\delta = 3$ and it is conjectured that it remains NP-hard for $\delta = 4$ as well. The problem is polynomial-time solvable when $\delta = 5$. In this paper, we show that, for any arbitrary collection of points in the plane, there always exists a degree-4 spanning tree of weight at most 1.1381, $(\sqrt{2}+2)/3$ to be exact, times the weight of a minimum spanning tree (MST). In particular, we present an improved analysis of Chan's degree-4 MST algorithm [4].

Previous results. Arora [1] and Mitchell [9] presented PTASs for TSP in Euclidean metric, for fixed dimensions. Unfortunately, neither algorithm extends to find degree-3 or degree-4 trees. Recently, Arora and Chang [3] have devised a quasi-polynomial-time approximation scheme for the Euclidean degree-δ spanning tree problem in $\Re^d$. As of now, there is no PTAS for finding spanning trees of degree 3 or 4 [2].

For points in the plane, Khuller et al [8] showed how to find degree-3 and degree-4 spanning trees whose weights are at most 1.5 and 1.25 times the weight of an MST, respectively. The degree 4 ratio was improved to 1.175 by Jothi and Raghavachari [6]. In an independent and parellel work, Chan [4] improved the ratio for degree-4 spanning trees to 1.143. He also improved the ratio for degree-3 spanning trees to 1.402, for points in the plane, using an elegant recursive algorithm.

In this paper, we present an improved analysis of Chan's degree-4 MST algorithm [4] thereby showing that, for an arbitrary collection of points in the plane, there always exists a degree-4 spanning tree of weight at most 1.1381, $(\sqrt{2}+2)/3$ to be exact, times the weight of a minimum spanning tree (MST). The difficulties in improving Chan's ratio was overcome by using a more careful charging scheme complemented by a new savings analysis. In addition, we show our ratio is tight and cannot be improved unless a more global approach is considered, instead of just local changes.

We first show that the angle enclosed between any two sides of a triangle can be used to bound the weight on the third side in a precise manner. Of course, the third side can be expressed exactly using trigonometry, but this formulation is unsuitable due to its non-linear nature. Our method provides a linear approximation. We then show that two MST edges intersecting at an acute angle force edge-weight constraints on each other, and this plays an important role in the improved analysis.

2 Degree-4 spanning trees

Let $|uv|$ be the Euclidean distance between u and v. Let $\angle ABC$ denote the angle formed at B between AB and BC. We start with a minimum spanning tree (MST) of graph G rooted at one of its leaf nodes. Our algorithm decreases the degree of high-degree nodes by local changes around it. Let x be a child of v in a tree T. Node x is defined to be a *biological* child of v if x is a child of v in the original MST, else it is a *foster* child.

We first note some interesting geometric properties, including that of MSTs in $\Re^d$. Due to lack of space, many proofs are omitted (see [7] for the full paper).

Lemma 1 *Let AB and BC be edges meeting at B. Let $x = |AB|$, $y = |AC|$, $z = |BC|$ and $\theta_1 = \angle ABC < 60°$. Let $z \geq y \geq x$. Then, for a fixed θ_1, $z - y$ is minimum when $x = y$.*

The following lemma proves an upper bound on the increase in weight when a node's degree is decreased in the usual way, in terms of the angle enclosed.

Lemma 2 ([4, 6]) *Let AB and BC be two edges incident on point B. Let $|AB| \leq |BC|$ and let $\theta = \angle ABC$. Then $|AC| \leq F(\theta)|AB| + |BC|$, where $F(\theta) = \sqrt{2(1-\cos\theta)} - 1 = 2\sin\frac{\theta}{2} - 1$.*

This lemma provides a better bound for the increase in the weight of the tree than just the triangle inequality. It can be verified that $|AC| \leq F(\theta)|AB| + |BC| \leq |AB| + |BC|$. We now prove that MST edges that intersect at a node, at an acute angle, force edge-weight constraints on each other.

*Research supported in part by the National Science Foundation under grant CCR-9820902.

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Lemma 3 *Let AB and BC be two edges that intersect at point B in an MST of set of points in $\Re^d$. Let $\theta = \angle ABC$. If $\theta < 90°$ then,*

$$2|BC|\cos\theta \leq |AB| \leq \frac{|BC|}{2\cos\theta}$$

Corollary 4 *Let AB and BC be edges meeting at B, and let AB be an MST edge and BC be a non-MST edge. Let $\theta = \angle ABC$. If $\theta < 90°$ then, $|BC| \geq 2|AB|\cos\theta$.*

Lemma 5 *Let V be a degree-5 node in an MST T of a set of points in $\Re^2$. Let P be its parent and A, B, C, and D be its children. Let the degree of V be decreased from 5 to 4 by replacing BV by AB, where $|AV| \leq |BV|$. Let $\angle AVB = \theta$. Let k of the children of V be at a distance of $|AV|$ or more from V. Then the increase in the weight of the tree is at most*

$$\frac{F(\theta)}{k}\Big(|AV| + |BV| + |CV| + |DV|\Big)$$

Therefore, the increase in weight can be "charged" to the k edges from V to its children, and the charge on each of these edges is at most $\frac{1}{k}F(\theta)$.

We first give a brief overview of Chan's algorithm [4] before proceeding to its approximation analysis.

Overview of Chan's algorithm. It recursively transforms the rooted tree T into a new degree-4 spanning tree with the inductive hypothesis that the root v of tree T has degree 3 in the new tree.

Let $\tau = 1.143$. Let T and T' be two subtrees, of an original MST, rooted at v and v', respectively. Let $T \nwarrow T'$ be a tree obtained by making v' a child of T. It recursively transforms $T \nwarrow T'$ to a new tree such that v has degree at most 3 in the new tree and the new tree has weight at most $|vv'| + \tau(w(T) + w(T'))$. It chooses a convenient permutation $v_1, \ldots, v_k$ of the k children of v in T together with v' (with $T_1, \ldots, T_k$ being their corresponding subtrees) for transformation.

Our analysis. Let v be the vertex under consideration whose degree has to be reduced. Let v have k biological children and at most 1 foster child. When $k \leq 3$, Chan showed that the ratio is bounded by $(\sqrt{2}+2)/3 < 1.1381$. We were able to improve Chan's ratio of 1.143 by tackling the case, $k = 4$, for which his analysis is tight. As per his induction hypothesis, v has a total of at most 5 children (4 biological and 1 foster). In essence, our objective is to reduce the degree of v from 5 to 3 (degree induced on v by its parent is excluded, but counts in the final solution which makes v's degree to be 4). The algorithm reduces v's degree from 5 to 3 by performing local changes around v.

To understand our analysis in a nutshell, consider Fig. 1 with v being the node whose degree we wish to reduce from 5 to 3, nodes v_1, v_2, v_3, v_4 being v's biological children, and v' being v's foster child. Suppose $\angle v_1vv' = \theta_5 \leq 60°$ (this is possible as vv' is a non-MST edge). Say, Chan's algorithm considers a transformation which involves replacing

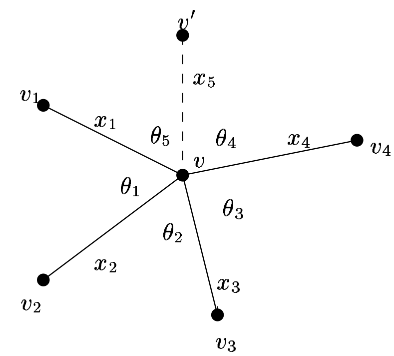

Figure 1: *Notation for $k = 4$ analysis.*

edges vv' with v_1v' and, say, vv_4 with v_3v_4. While Chan's analysis would directly charge the extra weight inolved in such a transformation to the MST edges involved, our analysis proceeds by calculating the potential savings due to the replacement of edge vv' by v_1v' (notice that $\theta_5 \leq 60°$ and $vv' \geq vv_1$ as vv_1 was chosen over v_1v' to be the MST edge) and use it to absorb part of the extra charge incurred due to the other replacement ($vv_4 \rightarrow v_3v_4$).

Given below is our analysis for the case $k = 4$. To make the description easier, we introduce a function called "Reduce".

Reduce(v, x, y): Let vx and vy be two edges incident on point v. Reduce(v, x, y) replaces the edge $\max\{vx, vy\}$ by xy. In simple terms, v's degree is reduced by 1, by donating one of $\{x, y\}$.

Let v_1, v_2, v_3, v_4 be the biological children of v in T and let v' be the foster child of v. Let v and its children be placed as shown in Fig. 1. Let $|vv_1| = x_1, |vv_2| = x_2, |vv_3| = x_3, |vv_4| = x_4, |vv'| = x_5, \theta_1 = \angle v_1vv_2, \theta_2 = \angle v_2vv_3, \theta_3 = \angle v_3vv_4, \theta_4 = \angle v_4vv'$ and $\theta_5 = \angle v'vv_1$. Since vv_1, vv_2, vv_3 and vv_4 are MST edges, $\theta_1, \theta_2, \theta_3, \theta_4 + \theta_5 \geq 60°$. Also, $\max\{\theta_1, \theta_2, \theta_3, \theta_4 + \theta_5\} \geq 120°$ considering the fact that one other MST edge, connecting v to its parent exists (not shown in figure). We consider three cases (the missing one is symmetric).

Case 1: $\theta_4 \leq 60°$ and $\theta_5 \leq 60°$. We handle this case in the same way as in [4]. Extra weight involved is bounded by 0.1331.

Case 2: $\theta_4 \geq 60°$ and $\theta_5 \leq 60°$. Since $\theta_5 \leq 60°$, $x_1 \leq x_5$ (otherwise $|v'v_1| < |vv_1|$, which contradicts the fact that vv_1 was chosen over $v'v_1$ to be an MST edge).

Case 2.1: $\theta_1 \geq 120°$ or $\theta_4 + \theta_5 \geq 120°$.

Call Reduce(v, v_1, v'). Since $\theta_5 \leq 60°$, no extra weight is incurred due to the call. By Lemma 2, we have permutations with extra weight bounded by

$$F(\theta_2)\min\{x_2, x_3\}, F(\theta_3)\min\{x_3, x_4\}.$$

Thus, the minimum extra weight is at most the smaller of the following values:

$$F(\theta_2)x_2, \min\{F(\theta_2), F(\theta_3)\}x_3, F(\theta_3)x_4.$$

Since the minimum is less than or equal to the harmonic mean, the minimum of these quantities is at most

$$\frac{1}{3}\text{H.M.}\{F(\theta_2), \min\{F(\theta_2), F(\theta_3)\}, F(\theta_3)\}(x_2 + x_3 + x_4).$$

Since $\theta_2 + \theta_3 \leq 180°$, the above coefficient is bounded by $\frac{1}{3}F(90°) = (\sqrt{2}+2)/3 < 0.1381$.

Case 2.2: $\theta_2 \geq 120°$ (Case $\theta_3 \geq 120°$ is symmetric).

Case 2.2.1: x_3 or x_4 is the smallest among $\{x_1, x_2, x_3, x_4\}$.

(2.2.1a) If $\theta_3 \leq 101.8°$, then call Reduce(v, v_1, v'). Since $\theta_5 \leq 60°$, no extra weight is incurred. Call Reduce(v, v_3, v_4). By Lemma 5, extra weight $F(\theta_3)\min\{x_3, x_4\}$ is charged to $\{vv_1, vv_2, vv_3, vv_4\}$ and is bounded by $0.1381(x_1 + x_2 + x_3 + x_4)$.

(2.2.1b) Else if $\max\{x_1, x_2, x_4\} \neq x_4$, then choose θ_1 and θ_4. Note that $\theta_1 + \theta_4 + \theta_5 \leq 138.2°$. Call Reduce$(v, v_1, v_2)$ and Reduce(v, v_4, v'). By Lemma 5, if $\theta_4 \leq 69.36°$, extra weights $F(\theta_1)\min\{x_1, x_2\}$ and $F(\theta_4)\min\{x_4, x_5\}$ are charged to $\{vv_1, vv_2\}$ and $\{vv_4\}$ respectively, else extra weights $F(\theta_1)\min\{x_1, x_2\}$ and $F(\theta_4)\min\{x_4 m, x_5\}$ are charged to $\min\{vv_1, vv_2\}$ and $\{\max\{vv_1, vv_2\}, vv_4\}$ respectively.

(2.2.1c) Else ($\max\{x_1, x_2, x_4\} = x_4$) if $\theta_4 \leq 69.36°$, then call Reduce(v, v_1, v_2) and Reduce(v, v_4, v_5). Since, $\theta_1 + \theta_4 + \theta_5 \leq 138.2°$ and $\theta_1, \theta_4 \geq 60°$, extra weights of at most $F(78.2°)\min\{x_1, x_2\}$ and $F(69.36°)\min\{x_4, x_5\}$ are charged to $\{vv_1, vv_2\}$ and $\{vv_4\}$, respectively (by Lemma 5), and is bounded by $0.1381(x_1 + x_2 + x_4)$.

(2.2.1d) Else $\theta_5 \leq 8.84°$. Hence $\theta_1 + \theta_5 \leq 68.84°$ and $\theta_4 + \theta_5 \leq 78.2°$. Call Reduce$(v, v_2, v')$ and Reduce(v, v_1, v_4). By Lemma 5, extra weights $F(\theta_1 + \theta_5)\min\{x_2, x_5\}$ and $F(\theta_4 + \theta_5)\min\{x_1, x_4\}$ are charged to $\{vv_2\}$ and $\{vv_1, vv_4\}$, respectively, and is bounded by $0.1381(x_1 + x_2 + x_4)$.

Case 2.2.2: x_3 or x_4 is 2nd smallest among $\{x_1, x_2, x_3, x_4\}$.

(2.2.2a) If $\theta_3 \leq 90°$, then call Reduce(v, v_1, v'). Since $\theta_5 \leq 60°$, no extra weight is incurred. Call Reduce(v, v_3, v_4). By Lemma 5, extra weight $F(\theta_3)\min\{x_3, x_4\}$ is charged to $\{vv_3, vv_4\}$ and the longest of $\{vv_1, vv_2\}$, and is bounded by $0.1381(x_1+x_2+x_3+x_4)$.

(2.2.2b) Else $\theta_1 + \theta_4 + \theta_5 \leq 150°$ and hence $\theta_5 \leq 30°$.

(2.2.2b-1) If $x_1 = \min\{x_1, x_2\}$, w.l.o.g. let $x_2 \leq x_4$. Since $\min\{\theta_1, \theta_4 + \theta_5\} \leq \frac{240° - \theta_3}{2}°$, by Lemma 3, $x_1 \geq 2x_2\cos(\frac{240° - \theta_3}{2})$. Call Reduce$(v, v_1, v')$. Since $\theta_5 \leq 30°$, no extra weight is incurred due to the call. Also, since vv_1 is an MST edge, $x_5 > x_1$ and thus, by Corollary 4, $x_5 \geq 2x_1\cos\theta_5$. By Lemma 1, $|vv'| - |v_1 v'|$ results in savings of at least $(2\cos\theta_5 - 1)x_1$. Let T_{before} be the subtree induced by nodes v, v_1, v_2, v_3, v_4 and v' and let T_{after} be the subtree induced by nodes v, v_1, v_2, v_3 and v_4. Clearly, as per our argument above, the weight of T_{after} is $(2\cos\theta_5 - 1)x_1$ less than that of T_{before}. Since our goal is to bound the extra weight, incurred during local transformations, to within 0.1381 times the MST weight, as per our charging policy, every MST edge e can be charged an extra weight of $0.1381e$. The savings obtained, due to the transformation from T_{before} to T_{after}, is equivalent to having atleast $\frac{2\cos 30° - 1}{0.1381}$ extra vv_1 edges, each of which can be charged $0.1381x_1$. In other words, it is as if we have at least an additional $(\frac{2\cos 30° - 1}{0.1381})vv_1$ to charge. Call Reduce(v, v_3, v_4). By Lemma 5, extra weight $F(\theta_3)\min\{x_3, x_4\}$ is charged to $\{vv_1, vv_2, vv_3, vv_4\}$ and $(\frac{2\cos 30° - 1}{0.1381})vv_1$, and is given by

$$\frac{F(\theta_3)(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 30° - 1}{0.1381}x_1)}{3 + 2\cos(\frac{240° - \theta_3}{2})\left(1 + \frac{2\cos 30° - 1}{0.1381}\right)}$$

which is bounded by $0.079(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 30° - 1}{0.1381}x_1)$.

(2.2.2b-2) Else ($x_1 \neq \min\{x_1, x_2\}$) the analysis proceeds in the same way as done in the previous step, except that the extra weight $F(\theta_3)\min\{x_3, x_4\}$ is charged to $\{vv_1, vv_3, vv_4\}$ and $(\frac{2\cos 30° - 1}{0.1381})vv_1$, and is given by

$$\frac{F(\theta_3)(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 30° - 1}{0.1381}x_1)}{3 + \frac{1}{0.1381}(2\cos 30° - 1)}$$

which is bounded by $0.048(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 30° - 1}{0.1381}x_1)$.

Case 2.2.3: $x_3, x_4 \geq x_1, x_2$.

(2.2.3a) If $\theta_3 \leq 79.29°$, Call Reduce(v, v_1, v'). Since $\theta_5 \leq 60°$, no extra weight is incurred due to the call. Call Reduce(v, v_3, v_4). By Lemma 5, extra weight $F(\theta_3)\min\{x_3, x_4\}$ is charged to $\{vv_3\}$ and $\{vv_4\}$, and is bounded by $0.1381(x_3 + x_4)$.

(2.2.3b) Else if $\theta_4 \leq 69.36°$ and $\theta_1 \leq 90°$, then call Reduce(v, v_4, v_5) and Reduce(v, v_1, v_2). By Lemma 5, extra weights $F(\theta_4)\min\{x_4, x_5\}$ and $F(\theta_1)\min\{x_1, x_2\}$ are charged to vv_4 and $\{vv_1, vv_2, vv_3\}$, respectively, and is bounded by $0.1381(x_2 + x_2 + x_3 + x_4)$.

(2.2.3c) Else if $\theta_4 \leq 69.36°$ and $\theta_1 > 90°$, then $\theta_5 \leq 10.71°$ and $60° \leq \theta_4 + \theta_5 \leq 70.91°$. Since $\theta_2 + \theta_4 + \theta_5 = 360° - \theta_1 - \theta_3 \leq 190.71°$, by Lemma 3, $x_1 \geq 2x_4\cos(190.71° - \theta_2)$. Call Reduce$(v, v_1, v')$. Since $\theta_5 \leq 10.71°$, no extra weight is incurred due to the call. Also, since vv_1 is an MST edge, $x_5 > x_1$ and thus, by Corollary 4, $x_5 \geq 2x_1\cos\theta_5$. By Lemma 1, $|vv'| - |v_1 v'|$ results in savings of at least $(2\cos\theta_5 - 1)x_1$. It is as if we have at least an additional $(\frac{2\cos 10.71° - 1}{0.1381})vv_1$ to charge. Call Reduce(v, v_2, v_3). By Lemma 5, extra weight $F(\theta_2)\min\{x_2, x_3\}$ is charged to $\{vv_1, vv_2, vv_3, vv_4\}$ and $(\frac{2\cos 10.71° - 1}{0.1381})vv_1$, and is given by

$$\frac{F(\theta_2)(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 10.71° - 1}{0.1381}x_1)}{3 + 2\cos(190.71° - \theta_2)\left(1 + \frac{2\cos 10.71° - 1}{0.1381}\right)}$$

which is bounded by $0.089(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 10.71° - 1}{0.1381}x_1)$.

(2.2.3d) Else ($\theta_4 > 69.36°$) $\theta_5 \leq 31.35°$.

(2.2.3d-1) If $\theta_5 \leq 11°$, then since $\theta_1 + \theta_4 + \theta_5 = 360° - \theta_3 - \theta_2 \leq 280.71° - \theta_2$ and $x_2 \leq x_4$, by Lemma 3,

$x_1 \geq 2x_2 \cos(\frac{280.71^\circ - \theta_2}{2})$. Call Reduce($v, v_1, v'$). Since $\theta_5 \leq 11^\circ$, no extra weight is incurred due to the call. Also, since vv_1 is an MST edge, $x_5 > x_1$ and thus, by Corollary 4, $x_5 \geq 2x_1 \cos\theta_5$. By Lemma 1, $|vv'| - |v_1v'|$ results in savings of at least $(2\cos\theta_5 - 1)x_1$. So, it is as if we have at least an additional $(\frac{2\cos 11^\circ - 1}{0.1381})vv_1$ to charge. Call Reduce(v, v_2, v_3). By Lemma 5, extra weight $F(\theta_2)\min\{x_2, x_3\}$ is charged to $\{vv_1, vv_2, vv_3, vv_4\}$ and $(\frac{2\cos 11^\circ - 1}{0.1381})vv_1$, and is given by

$$\frac{F(\theta_2)(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 11^\circ - 1}{0.1381}x_1)}{3 + 2\cos(\frac{280.71^\circ - \theta_2}{2})\Big(1 + \frac{2\cos 11^\circ - 1}{0.1381}\Big)}$$

which is bounded by $0.13(x_1+x_2+x_3+x_4+\frac{2\cos 11^\circ - 1}{0.1381}x_1)$.

(2.2.3d-2) Else if $11^\circ < \theta_5 \leq 25^\circ$, then since $\theta_1 = 360^\circ - \theta_2 - \theta_3 - \theta_4 - \theta_5 \leq 200.35 - \theta_2$, by Lemma 3, $x_1 \geq 2x_2\cos(200.35^\circ - \theta_2)$. Call Reduce($v, v_1, v'$). Since $\theta_5 \leq 25^\circ$, no extra weight is incurred due to the call. Also, since vv_1 is an MST edge, $x_5 > x_1$ and thus, by Corollary 4, $x_5 \geq 2x_1\cos\theta_5$. By Lemma 1, $|vv|' - |v_1v'|$ results in savings of at least $(2\cos\theta_5 - 1)x_1$. So, we have an additional $(\frac{2\cos 25^\circ - 1}{0.1381})vv_1$ to charge. Call Reduce(v, v_2, v_3). Using Lemma 5, the extra weight $F(\theta_2)\min\{x_2, x_3\}$ is charged to vv_1, vv_2, vv_3, vv_4 and $(\frac{2\cos 25^\circ - 1}{0.1381})vv_1$, and is given by,

$$\frac{F(\theta_2)(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 25^\circ - 1}{0.1381}x_1)}{3 + 2\cos(200.35^\circ - \theta_2)\Big(1 + \frac{2\cos 25^\circ - 1}{0.1381}\Big)}$$

which is bounded by $0.138(x_1+x_2+x_3+x_4+\frac{2\cos 25^\circ - 1}{0.1381}x_1)$.

(2.2.3d-3) Else ($25^\circ < \theta_5 \leq 31.35^\circ$), since $\theta_1 = 360^\circ - \theta_2 - \theta_3 - \theta_4 - \theta_5 \leq 186.35 - \theta_2$, by Lemma 3, $x_1 \geq 2x_2\cos(186.35^\circ - \theta_2)$. Call Reduce($v, v_1, v'$). Since $\theta_5 \leq 31.25^\circ$, no extra weight is incurred due to the call. Also, since vv_1 is an MST edge, $x_5 > x_1$ and thus $x_5 \geq 2x_1\cos\theta_5$. By Lemma 1, $|vv'| - |v_1v'|$ results in savings of at least $(2\cos\theta_5 - 1)x_1$. So, it is as if we have at least an additional $(\frac{2\cos 31.35^\circ - 1}{0.1381})vv_1$ to charge. Call Reduce(v, v_2, v_3). Using Lemma 5, the extra weight $F(\theta_2)\min\{x_2, x_3\}$ is charged to $\{vv_1, vv_2, vv_3, vv_4\}$ and $(\frac{2\cos 31.35^\circ - 1}{0.1381})vv_1$, and is given by

$$\frac{F(\theta_2)(x_1 + x_2 + x_3 + x_4 + \frac{2\cos 31.35^\circ - 1}{0.1381}x_1)}{3 + 2\cos(186.35^\circ - \theta_2)\Big(1 + \frac{2\cos 31.35^\circ - 1}{0.1381}\Big)}$$

which is bounded by $0.1(x_1+x_2+x_3+x_4+\frac{2\cos 31.35^\circ - 1}{0.1381}x_1)$.

Case 3: $\theta_4 \geq 60^\circ$ and $\theta_5 \geq 60^\circ$. The proof is similar to that of Case 2, and due to lack of space, it is omitted.

Theorem 6 *For any arbitrary collection of points in the Euclidean plane, there always exists a degree-4 spanning tree of weight at most $(\sqrt{2} + 2)/3$ times the weight of an MST.*

3 Conclusion

By presenting an improved approximation analysis for Chan's degree-4 MST algorithm, we showed that, for any arbitrary collection of points, there always exists a degree-4 spanning tree of weight at most 1.1381 times the weight of an MST. Our ratio for degree-4 spanning trees cannot be improved unless a more global approach is considered, instead of just the local changes that we considered in this paper, as there exists placement of points for the case $k = 3$, such that doing local changes alone does not reduce the ratio. There exists degree-4 and degree-3 trees (regular pentagon and square with an extra point at the center) whose weights are at most $\frac{2\sin 36^\circ + 4}{5}$ and $\frac{\sqrt{2}+3}{4}$ times the weight of an MST, respectively. It should be interesting to know whether better approximation algorithms can be developed to achieve ratios anywhere close to these lower bounds.

Acknowledgments. We thank Timothy Chan, Ovidiu Daescu and R. Chandrasekaran for valuable discussions.

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