Decomposition of planar motions into reflexions and rotations with distance constraints

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Abstract

Three variations are given on a theorem of Romerio and Burckhardt stating that under the constraint of a minimum turning radius the number of movements needed to re-position an automobile-like vehicle is 3. In a more abstract geometric framework, vehicle positions are represented by tuples of points on which a transformation group G acts regularly, and questions of factorization under certain constraints are posed in G.

1 Introduction and overview

Every rigid planar motion (isometry of the Euclidean plane) can be decomposed into a product of at most 3 reflections, and the bound 3 is best possible. The basic theory of isometries also tells us that every direct rigid motion (orientation-preserving isometry) can be decomposed into a product of at most 2 rotations, and here again the bound 2 is obviously best possible - non-trivial translations need 2 rotations.

These decompositions are not unique: in fact any product of 2 or 3 reflections can be represented in infinitely many ways as a product of 2 or 3 reflections, and if any particular reflection is forbidden, a representation so constrained will still exist. This can be viewed as a consequence of the classical three-reflections theorems for concurrent and parallel mirrors. (For background see e.g. Ryan [4].) Even stronger, any finite or even countably infinite set of reflections can be forbidden, and representations by 2 or 3 reflections will continue to exist. Broader restrictions on what reflections may be used in representing a motion may lead either to non-existence of representations or to the requirement of a higher number of reflections (4 or more) needed to represent a motion.

The problem of decomposing direct isometries of the plane into a minimum number of constrained rotations was studied by Romerio and Burckhardt [3]. The constraints they consider correspond to the motion of automobile-like vehicles that move in discrete phases, each phase of movement consisting of the setting of the steering wheel in a fixed position after which the vehicle travels along the circular trajectory (or straight line) determined by the fixed position of the wheel. Moreover, the wheel cannot be turned too sharply. Vehicle positions can be modelled as couples of points (A, B) at unit distance. The group of direct isometries acts regularly on the set of such couples. The contraint on vehicle movement during a phase is modelled by requiring the transformation to be a rotation whose center is on the line drawn through A and perpendicular to AB, moreover the center of rotation is required to be at a distance more than a prescribed minimum from A. The number of such restricted motions required to map one vehicle position to another was shown to be 3 in [3], even if translations along AB are allowed. We show that two different relaxations of the constraint considered by Romerio and Burckhardt lead to decompositions into at most 2 rotations. For arbitrary isometries, direct or indirect, we also give a theorem of decomposition into a product of reflections, subject to the constraint that the distance of the reflection mirror be at least a prescribed minimum from the position of the object to be moved - but now the object is no longer thought of as an automobile-like vehicle and its position is modelled by a triple of points (A, B, C) mutually at unit distance from each other. The group of all plane isometries acts regularly on the set of all such triples, and this group is generated by reflections, any isometry being the product of at most 3 reflections. Proposition 3 below asserts that under the mirror distance constraint, the number of reflections needed to map any point triple to another given triple is 4.

2 Formal definitions and decomposition theorems

The following framework is not the most general possible, but it is general enough for the results of Romerio and Burckhardt and the decomposition theorems of this paper.

Let G be a group acting regularly on a set V and let H be a non-empty set of generators of G closed under taking of inverses. (This implies in particular that every member of G can be factorized into a product of some positive number of members of H.) Considering also the action of G on itself by conjugation, a natural action of G on GxV is given by

$$f(g, v) = (fgf^{-1}, fv)$$

A relation R between G and V (i.e. subset of GxV) that is closed under this action is called a *constraint*. Members of G are called *transformations*, those of V *positions*, and if a couple (g, v) is in R then we say that the transformation g is *allowed* by the position v. If there is a positive integer m such that for every v in V every g in G can be written as a product of at most m (not necessarily distinct) members of H, $g = h_m \dots h_1$, with the property that for each

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$i = 1, \ldots, m$ the transformation h_i is allowed by position $h_{i-1} \ldots h_1(v)$, then we say that m is the *minimum number of factors in constrained decompositions of members of* G (*into products of members of* H). Clearly such m may fail to exist, and it depends not only on the choice of H but on the specification of the constraint R as well.

Two classes of extreme situations are when R is empty or the full relation GxV. Trivially, if R is empty then there is no minimum number of factors. The case RxV is non-trivial, even though here the specification of V and of the group action is irrelevant. This case can be called the *unconstrained* case. For example, with G the group of plane isometries and H the set of reflections, we have $m = 3$. With G the group of direct isometries of the plane and H the set of all rotations, we have $m = 2$. The results of Romerio and Burckhardt [3] can now be restated as follows:

Let positions be defined as couples (A, B) of points in the plane at unit distance, $d(A, B) = 1$, and let G be the group of direct isometries.

(i) *Let H be the set of rotations. Let $M \geq 0$. Let each position (A, B) allow every rotation whose center is on the line perpendicular to AB drawn through A and at a distance at least M from A. Then $m = 3$.*

(ii) *Let H be the set of rotations plus translations. Let each position (A, B) allow the same rotations as in (i) and also all translations along the line AB. Then $m = 3$.*

Note that the unconstrained versions of (i) and (ii) give $m = 2$ and $m = 1$, respectively. We give three further constrained decomposition theorems. In the first two, G is the group of direct plane isometries , the positions are couples (A, B) of points at distance $d(A, B) = 1$, H is the set of rotations, and the constraints are two different partial relaxations of the constraint considered by Romerio and Burckhardt as described in (i) above.

Proposition 1 *Let positions be defined as couples of points (A, B) at distance $d(A, B) = 1$, and let each position (A, B) allow every rotation whose center is on the line through A perpendicular to AB. Then the minimum number of factors in constrained decompositions of direct plane isometries into rotations is $m = 2$.*

Sketched proof. Observe first that given a position (A, B) each circle with center X tangent to AB at A is contained in one or the other half-plane determined by the line AB, according to whether the triple (A, B, X) is clockwise or counter-clockwise.

Suppose a direct isometry g and a position (A, B) are given. If g fixes A then g is a rotation allowed by (A, B). Otherwise draw sufficiently small disjoint circles Q and K tangent to AB at A and to $gAgB$ at gA, respectively, say with respective centers X and Y, and such that both circles are contained in the same half-plane determined by the line AB.

If one of the triples (A, B, X) and (gA, gB, Y) is clockwise and the other is counter-clockwise, then inflate Q with center of dilatation A until it becomes a circle Q' tangent to K at a point T, the interiors of the circles being still disjoint. The transformation g is then decomposed into a rotation about the center of Q' bringing A to T, followed by a rotation about the center Y of K bringing T to gA.

If both (A, B, X) and (gA, gB, Y) are clockwise or both are counter-clockwise, then further inflate Q until it becomes tangent to K and includes K, and define the two rotations similarly. Δ

Proposition 2 *Let positions be defined as couples of points (A, B) of the plane, let $M \geq 0$, and let each position (A, B) allow every rotation whose center is at a distance at least M from A. Then the minimum number of factors in constrained decompositions of direct plane isometries into rotations is $m = 2$.*

Sketched proof. Observe first that given positions (A, B) and (A', B') and a point X, there are exactly two rotations h around X such that the line $hAhB$ is parallel to $A'B'$ and just one of these rotations is such that the directed segments $hAhB$ and $A'B'$ have opposite directions.

Suppose a direct isometry g and a position (A, B) are given. By appropriate choice of a rotation h with center X far enough from A, we can rotate (A, B) to a position (hA, hB) such that hA is far enough from gA and the directed segments $hAhB$ and $gAgB$ are parallel with opposite directions. The let f be the half-turn about the middle point of the segment $gAhA$. We have $g = fh$. Δ

In the following proposition the group G is the group of all plane isometries, direct and indirect. This group acts regularly on the set of triples of points (A, B, C) forming an equilateral triangle of unit side length, $d(A, B) = d(A, B) = d(B, C) = 1$. As generating set H we take all reflections. Recall that the unconstrained minimum number of factors is $m = 3$.

Proposition 3 *Let positions be defined as triples of points (A, B, C) with $d(A, B) = d(A, C) = d(B, C) = 1$. Let $M \geq 0$. Let each position (A, B, C) allow every reflection whose mirror is at distance at least M from A. Then the minimum number of factors in constrained decompositions of plane isometries into reflections is $m = 4$.*

Sketched proof. Suppose an isometry g and a position (A, B, C) are given. We indicate the construction of the reflection factors under the assumption that AB is not parallel to $gAgB$. In the parallel case special arguments of particular simplicity can be applied. Under the stated assumption, first we map (A, B, C) to (A', B', C') so that

(i) A' is far enough from the line $gAgB$,

(ii) the orthogonal projection of A' on the line $gAgB$ is also far enough from gA,

(iii) and the directed segment $A'B'$ is parallel with, and opposite in direction to, the directed segment $gAgB$.

This we do with a single reflection in a mirror far enough from A if g is a direct isometry, otherwise we do it with two successive reflections in mirrors sufficiently distant. Then we reflect in a mirror parallel to $gAgB$ and at half distance between the lines $gAgB$ and $A'B'$, followed by an appropriate reflection in a mirror perpendicular to the line $gAgB$, to bring A' to gA. The optimality of $m = 4$ is shown by letting g be a rotation by $\pi/2$. Δ

We note that the proof of Proposition 1 involves an inflating process which can be viewed as a time-parametrized continuous process that is applied until a certain geometric configuration is achieved at the limit. In [3] the results are established both using linear algebra and by a ruler-and-compass argument. In fact the latter can also be replaced by a continuity argument, giving a somewhat different insight. Such replacement of discrete ruler-and-compass constructions by continuity arguments have shown their usefulness in other areas of polygon geometry [1, 2], and this approach may be applied to further instances of the general constrained decomposition problem.

References

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