Approximating optimal paths in terrains with weight defined by a piecewise-linear function

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Abstract

Finding optimal paths in non-homogeneous terrains is a class of problem that presents itself in many situations. One of the best-known versions is the so-called Weighted Region Problem (WRP) [2],[3],[4],[5], based on a model of space with regions of constant weight. Here this version is generalized using linear functions defined to coincide with the weights assigned at the vertices of a patchwork of triangles, and a method is proposed of approximating the optimal path by a polygonal curve in $O(n^3)$ time. Experiments show that, despite its apparent complexity, this kind of problem can be solved by methods similar to those used for the WRP and at a computational cost of $O(n^2)$ in practice for models with more than 4000 regions.

1 Introduction

Let us consider a two-dimensional space domain consisting of a triangular grid where, at vertex V_i, there is a known weight ($w_{Vi} > 0$). Within each triangular region.R_j, a linear interpolating weight function $w_{Rj}(x, y)$ can be defined and this piecewise-linear function can be used over the domain as an approximation to the sampled weight function (Figure 1).

An optimal path between two points of this space will be a sequence of, in principle, tangent catenary arcs and straight segments. In regions with a weight gradient, the solution will contain catenary arcs and, occasionally, straight segments along the boundary. In regions of constant weight (three vertices with equal weight), the respective segment will be straight. The path may cross all regions more than once. However, given the exceptional conditions under which multiple crossings can occur, it can be assumed that the number of such crossings will be limited in a fairly regular triangular patchwork. Then, if the patchwork has n regions, the number of segments of the optimal path can be considered to be $O(n)^1$.

The model can be extended to cases with discontinuities, taking arbitrary linear functions for the weight on each patch. This makes it possible to consider the WRP as a particular case with a consequential loss of continuity of the slope at break points

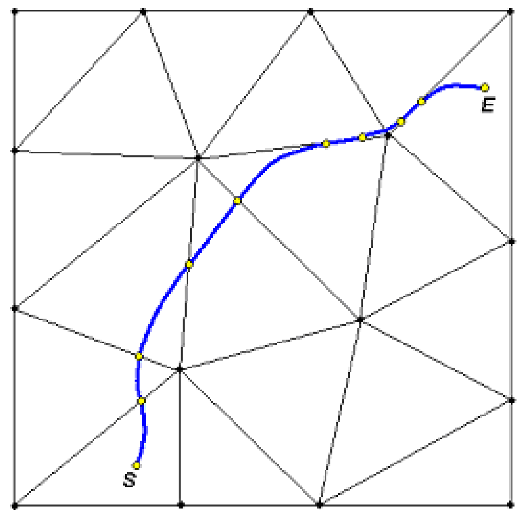

Figure 1: Optimal path with weight defined by a piecewise-linear function.

2 Approximation of the optimal path by a polygonal curve

The principal difficulty in identifying a complete solution in this model (as for the WRP), is to determine the sequence of patches through which the solution passes. But to reach this point, the whole domain needs to be explored, running tests at a heavy computational cost. The WRP has the advantage of the optimal path being a polygonal curve with segments wholly contained in the regions (provided they are convex) and of the ease of evaluation of the cost. In the present case, calculating the cost of crossing a particular patch involves first determining the catenary parameters of each segment. Matters are further complicated by possible patch re-entry after exit.

Under these circumstances, the question is whether the catenary arcs can be substituted by straight segments, that is, whether an approximate polygonal solution with vertices on the boundaries is acceptable. If so, this new problem is similar to the WRP and can be dealt with using similar techniques.

Let us focus on a catenary arc of the optimal path contained in a triangular region whose end points P_0 and P_1 lie

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[1]Mitchell and Papadimitrou [4] calculated the number of segments of the optimal path in WRP to be $O(n^2)$, based on a paradoxical example. In the context of our work, the grid obeys the need to interpolate the weight function and we understand that Delaunay triangulation meets the above-mentioned conditions of regularity, except in very exceptional cases that could be avoided by introducing some additional sampling points

on its boundary. This arc will be the optimal path between the two points, that is, the path minimizing the functional

$$C = \int_S w(x(s), y(s))ds \quad (1)$$

when $w(x, y)$ is a linear function. It is well known that (1) has another possible solution (Goldschmidt's solution), consisting of two straight segments leading from each point to the line $w(x, y) = 0$ and a segment of zero cost along this line [1],[6]. Consequently, the arc will either be better than Goldschmidt's solution or the result of an optimization with constraints where one of the boundaries obstructed Goldschmidt's solution. This means that it will be tangent to one of the end points of the catenary.

Let us suppose now the following strategy to replace the catenary arc by straight segments: a) if the catenary is better than a Goldschmidt solution, replace it by the segment P_0P_1, b) if not, replace it by two segments P_0Q and P_1Q tangent to its end points (Figure 2).

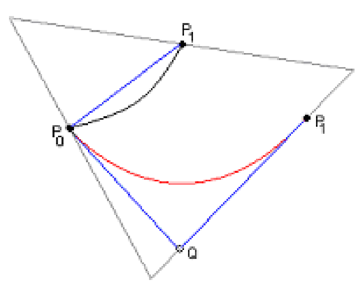

Figure 2: Strategies to approximate catenary arcs.

In both cases, a ratio can be calculated between the path costs in order to find out the relative error. The choice of coordinate axes is immaterial, as it is the choice of scale, given the linearity of the weight function. Hence it is sufficient to analyze the ratio functions for a simple problem with weight function $w(x, y) = y$, start point located at (0,1) and end point at any $P(x, y)$, with $x \geq 0$ and $y \leq 1$.

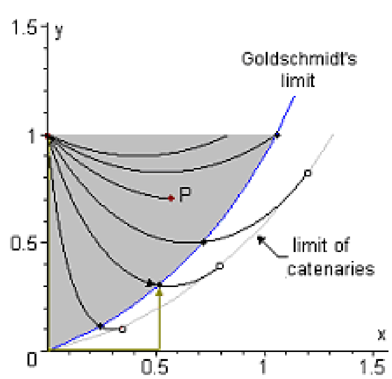

Figure 3: Restricted domain for the ratio analysis

In case a) (Figure 3) a ratio function $R(a, x)$ can be calculated for any point P of a catenary arc of parameter a starting at (0,1) and ending at the point where the catenary and Goldschmidt solutions have equal cost (Goldschmidt's limit). This function has a maximum $R = 1.055$ (Figure 4).

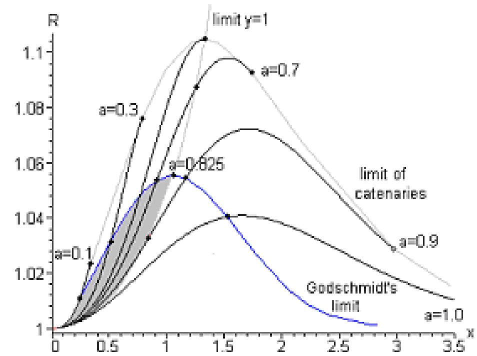

Figure 4: Ratio between the costs of paths in case a).

Proceeding as above, we can obtain a ratio $R^*(a, x)$ in case b) for any point P of a catenary arc of parameter a ending at the envelop of the family of catenaries starting at (0,1) (limit of catenaries). The maximum of the function is $R^* = 1.058$ (Figure 5).

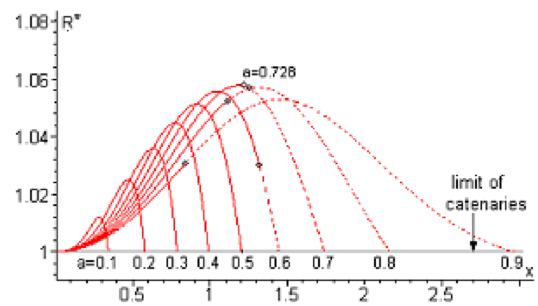

Figure 5: Ratio between the cost of paths in case b).

It follows that we can construct a polygonal with vertices on the region boundaries and a cost

$$C_R \leq C_c (1 + \beta)$$

where $\beta \leq 0.058$ and C_c is the real cost.

Furthermore, the size of the grid will influence the value of β, since with smaller triangular regions we will have small catenaries, within Goldschmidt's limit, that will be better approximated by means of straight segments. Consequently, it is reasonable to reformulate the original problem as finding the optimal polygonal path across a triangular patchwork with vertices lying on the region boundaries and with a cost expressed as

$$C_R = \sum_{k=1}^{n} w_k L_k$$

where w_k is the average of the weights at the ends of the segment and L_k is the segment length, as results from (1) following this path.

3 Algorithm

The chosen method, like approaches adopted by other authors, is based on a discretization of the edges of the triangular patches and introduces a certain number of new points (Steiner points) on these edges. With this discretization, we have a graph G in which each arc is assigned a weight that is the average of the end point weights (Figure 6). The problem can now be solved using Dijkstra's algorithm or any of its variants. Without loss of generality, the original start point (s) and end point (e) can be assumed to be vertices of the original patchwork and, therefore, nodes of the graph. If this were not the case, it would suffice to subdivide the region containing the start and end points into another three.

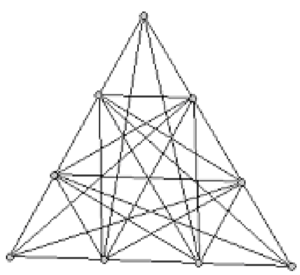

Figure 6: Subgraph connecting vertices and Steiner points in a patch.

The algorithm begins by identifying a first approximation to the optimal path that is accurate enough to determined a sequence of crossed patches and boundaries. To this end, Steiner points are introduced and the graph G is built. The optimal path $\pi'(s,e)$ in G is an approximation of the optimal polygonal path $\pi(s,e)$ in P, where the vertices match the graph nodes. A straightforward way to introduce the Steiner points is to divide the boundaries at regular intervals into a given number of segments. This method can be improved by setting a quantity ΔL such that each of the subdivisions comes as close as possible to, without exceeding, this quantity or, alternatively by setting the maximum difference Δw between weights of neighboring points. The accuracy of the chosen system will depend on the maximum values of the two parameters.

When the approximation $\pi'(s,e)$ to the optimal path is found, the algorithm enters a refinement phase in which the problem is constrained to the sub-domain of P formed by the crossed patches. At this stage, consideration is given to subdividing the remaining patches and increasing the number of Steiner points on them, with the effect of both improving the position of boundary crossing points and smoothing the representation of catenary arcs. These procedures are applied iteratively with the aim of reducing total path cost. This does not necessarily mean that the solution obtained is closer to the optimum. This will depend on the accuracy of the search process in the first phase, that is, the number of Steiner points used. If the sequence of patches crossed is determined correctly in the first phase, then the second phase does indeed produce a better (and smoother) solution.

Clearly, the greater the number of Steiner points, the more precise the approximation will be, while execution time will increase at the same. Therefore, a ratio between the number of Steiner points, the error estimate, and the time taken to solve the problem should be established. The reasoning used by Lanthier [3] to justify his approximate path search algorithm in weighted regions is used here to highlight the similarities and differences between this approach and the WRP[2].

Proposition 1 *A segment of the optimal path s_j within a patch R_j is approximated by another s'_j, the ends of which are neighboring points of the subdivision, such that*

$$C(s'_j) \leq C(s_j) + 2w_{Rj}\Delta L$$

where w_{Rj} is the maximum value of the weight function in R_j.

Lemma 2 *If $\pi(s,e)$ is an optimal path in P, there exists an approximate path $\pi'(s,e)$,in G such that*

$$C(\pi') \leq C(\pi) + 2mw_{\max}\Delta L$$

where $C(\pi')$is the cost of the approximate path, $C(\pi)$ the cost of the optimal path, m is the number of segments in each, and $w_{\max}$ is the maximum value of the weight function in the domain.

Theorem 3 *There is an approximation $\pi\prime(s,e)$ to the optimal path $\pi(s,e)$ of straight segments such that*

$$C(\pi') \leq C(\pi) + 2w_{\max}L_e$$

where L_e is the length of the longest boundary of any patch. Moreover, the computational cost of this approximation is $O(n^3)$, where n is the number of triangular patches.

4 Experimental results

To validate our proposal, a prototype was developed to check the effect of the algorithm parameters (the number of Steiner points and the level of refinement of the solution) on accuracy and computation time. As, to the best of our knowledge,

[2]We omit the details of the demonstrations due to space constraints on this version

there are no similar approaches, the results were compared with those of a traditional raster method using a fine rectangular grid.

All the tests were based on a map of a geographical terrain over which a variable number of points were chosen to measure, on a scale of 1 to 10, the cost of travel as affected by relief, vegetation and other factors. A Delaunay triangulation was set up over the sample points, and terrain models were generated with 120, 480, 1080, 1920, 3000 and 4320 triangular patches, assigning the calculated weights to the vertices. Problems were set up for each terrain model with start and end points situated at the outer regions to assure that the terrain was crossed in the path. The paths were finally checked for validity (orthogonality) against contours of the cost surface generated on raster models. The conclusions presented below are taken from the mean values of cost and execution time measured on each model with a range of parameter values.

Accuracy of the paths. The evolution of the calculated path cost as a function of the number of Steiner points was asymptotic, reaching a limit value at 20 points and more. With respect to the refinement of the trajectory by successive subdivision of the patches crossed in the initial solution the improvement in the approximate cost was less than might be expected. The indications are that, apart from exceptional cases, the utility of the refinement phase is to improve the form of the path rather than to bring about an effective reduction in cost, and a single level of refinement therefore appears to be sufficient.

Computation times. Computation times increased quadratically with the number of Steiner points (and so with the total number of nodes in the graph) in all models. The increase was slightly more marked in the refinement phase where the effect of proliferating Steiner points combined with the subdivision of patches. A more detailed analysis, with the model of 4320 patches, revealed that the computation time in the first phase was $O(n^2)$, whereas it was closer to $O(n^2 log(n))$ with three levels of subdivision. This can be explained by the fact that the number of nodes in the graph grows quickly at three subdivisions and is nearly $O(n^2)$, whereas the number of edges does not reach $O(n^3)$.

Comparing our model with the WRP model. While experimentation proceeded with the proposed model, tests were also carried out on models with patches of constant weight (averaged over each patch). The results revealed in some cases notable discrepancies in the trajectory of the optimal paths and their costs In general, the experiments indicate that the discrepancies are larger in models with sharp changes of weight and fewer patches. This is easy to understand in terms of inadequate interpolation, since both methods must give convergent results as the number of patches increases. This tendency supports the hypothesis that, in some real applications, the proposed model requires fewer patches than the WRP, and that this compensates for its greater complexity.

5 Conclusions

In this paper we have proposed a generalization of the Weighted Region Problem. This new point of view, it is hoped, will contribute to solving some real practical problems and, at the same time, open up alternative lines of research in this field.

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