A Stronger Version of Bárány's Theorem in the Plane *

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Abstract

We prove that given any 3 sets S_1, S_2 and S_3 of points in the plane each containing a given point $Z \in \mathbb{R}^2$ in its convex hull, we can always find 2 disjoint sets $T_1, T_2 \subset S_1 \cup S_2 \cup S_3$ each of which contains Z in its convex hull, such that $|T_i \cap S_j| \leq 1$ for $i = 1, 2$ and $j = 1, 2, 3$. This strengthens the two dimensional version of a theorem of I. Bárány.

1 Introduction

A theorem of I. Bárány [1, 2] states that given $d + 1$ sets of points $S_1, \ldots, S_{d+1} \subseteq \mathbb{R}^d$ and a point $Z \in \mathbb{R}^d$, such that $Z \in \operatorname{conv} S_i$, for $i = 1, \ldots, d + 1$, there exist points $v_1 \in S_1, \ldots, v_{d+1} \in S_{d+1}$ such that $Z \in \operatorname{conv}\{v_1, \ldots, v_{d+1}\}$. When $d = 2$, we shall prove the following stronger statement.

Theorem 1 *Given 3 sets of points $S_1, S_2, S_3 \subseteq \mathbb{R}^2$ and a point $Z \in \mathbb{R}^2$, such that $Z \in \operatorname{conv} S_i$, for $i = 1, 2, 3$, there exist* distinct *points $v_1, v_1' \in S_1$, $v_2, v_2' \in S_2$, and $v_3, v_3' \in S_3$ such that $Z \in \operatorname{conv}\{v_1, v_2, v_3\} \cap \operatorname{conv}\{v_1', v_2', v_3'\}$.*

Furthermore, we shall give an example, for $d = 3$, showing that an analogue of Theorem 1 does not hold in general.

2 Proof of Theorem 1

First, we may assume, without loss of generality, that the points of the sets S_1, S_2, S_3 are located on the unit circle, by projecting these points on the circumference of the circle.

Second, by Carathéodory Theorem (see [3]), we may assume, without loss of generality, that the cardinality of each set S_i is at most 3, for $i = 1, 2, 3$. In fact, we can further assume that $|S_i| = 3$, for $i = 1, 2, 3$, since we can extend S_i with a copy of an arbitrary point in S_i if $|S_i| = 2$. Thus, we assume in the rest of this section that we are given three triangles whose vertices lie on the circumference of a circle, such that each of them contains the center of the circle in its interior. For convenience, let us assume that the three vertices of each triangle are colored with a distinct color (red, blue and green). In other words, the vertices of one triangle are red, those of the second are blue, and those of the third are green. Thus, what we need to show is that, out of these nine vertices, we can get two triangles each formed by three vertices of different colors, such that each of these two triangles contains the center of the circle in its interior.

Consider an arbitrary diameter of the circle. Lets name the vertices of the three given triangles by the initials of their colors, and moreover name the vertices above the diameter with capital letter (R, B and G) and those below the diameter with small letters ($r, b,$ and g). Next, replace each point below the diameter with an opposite point above the diameter (i.e. the other point on the circumference that lies on the diameter formed by this point). We call this point the image of the first one. Let $\mathcal{S}$ be the sequence of the resulting nine points, sorted by the angle between the radius formed by each of these points and the selected diameter. Since each of the three triangles contains the center of the circle in its interior, it follows that the sequence $\mathcal{S}$ contains as a subsequence, for each of the three colors, either one small initial between two capital initials or one capital initial between two small initials. For example, for the red triangle, we have either the subsequence R, r, R or the subsequence r, R, r. Assume, without loss of generality, that $\mathcal{S}$ contains more capital initials than small initials. It follows that there are either 6 capitals and 3 smalls (we call such a sequence the $(6, 3)$-sequence), or 5 capitals and 4 smalls (we call such a sequence the $(5, 4)$-sequence). We may assume, without loss of generality, that $\mathcal{S}$ contains the subsequence R, r, R, the subsequence B, b, B, and either the subsequence G, g, G or g, G, g. One way, to prove our theorem, is to enumerate all the $\frac{9!}{3!3!3!}$ possibilities for the $(6, 3)$-sequences, and the $\frac{9!}{3!3!3!}$ possibilities for the $(5, 4)$-sequences, for a total of 3360 possibilities. Next, we give a precise proof that the theorem works, by reducing these cases to a small number of enumerations. For each of the possibilities, we get two triangles the vertices of each satisfying the above requirements (three different colors; one small between two capitals or one capital between two smalls).

A. The (6,3)-sequences

From the above discussion, it follows that any $(6, 3)$-sequence begins and ends with capital initials. Assume, without loss of generality, that it begins with R.

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case 1. The sequence $\mathcal{S}$ ends with R:

If b comes before g in $\mathcal{S}$, the two claimed triangles are R, b, G (this G comes after g and hence after b) and B, g, R (this B comes before b and hence before g). By symmetry, if g comes before b, the two triangles are R, g, B and G, b, R.

case 2. The sequence $\mathcal{S}$ does not end with R:

We may assume, without loss of generality, that $\mathcal{S}$ ends with a B. Assume that the first triangle is R, g, B (the first R and the last B). It follows that what is left from $\mathcal{S}$ is three subsequences: the subsequence r, R, the subsequence B, b, and the subsequence G, G. There are 6 possibilities putting the first two together:

1. B, r, b, R.
2. B, b, r, R.
3. r, R, B, b.
4. r, B, R, b.
5. B, r, R, b.
6. r, B, b, R.

For the first 4 possibilities, adding G anywhere in the subsequence, we can show, by enumeration, that we can always find the second triangle. For example, adding G to the first subsequence B, r, b, R, the possibilities are: the subsequence G, B, r, b, R with triangle G, b, R, the subsequence B, G, r, b, R with triangle G, b, R, the subsequence B, r, G, b, R with triangle G, b, R, the subsequence B, r, b, G, R with triangle B, r, G, and the subsequence B, r, b, R, G with triangle B, r, G.

For the last two subsequences: When G comes before B or r in B, r, R, b, or after b or R in r, B, b, R, we will not be able to get the second triangle. For such cases, there are 8 possibilities for $\mathcal{S}$. We enumerate these cases together with the corresponding triangles in Table 1.

	$\mathcal{S}$	Tri. 1	Tri. 2
1	$R, G, g, B, G, r, R, b, B$	R, g, B	G, r, B
2	$R, G, g, G, B, r, R, b, B$	R, g, B	G, r, B
3	$R, G, B, g, G, r, R, b, B$	B, g, R	G, r, B
4	$R, B, G, g, G, r, R, b, B$	B, g, R	G, r, B
5	$R, r, B, b, G, R, g, G, B$	R, g, B	R, b, G
6	$R, r, B, b, R, G, g, G, B$	R, g, B	R, b, G
7	$R, r, B, b, G, g, R, G, B$	B, g, R	R, b, G
8	$R, r, B, b, G, g, G, R, B$	B, g, R	R, b, G

Table 1: The sequence $\mathcal{S}$ and the corresponding two claimed triangles.

Note that the sequences $5-8$ are obtained from the sequences $1-4$ by reversing each sequence, and swapping every R with B and every r with b.

B. The (5,4)-sequences

Define an alternating $(5,4)$-sequence to be a sequence with 5 capitals and 4 smalls that alternate with each other. More precisely, in such a sequence, every small immediately follows a capital and immediately precedes another capital. The following lemma implies that unless a $(5,4)$-sequence is alternating, we can convert it into a $(6,3)$-sequence.

Lemma 2 *Given a $(5,4)$-sequence that is non-alternating, we may change the corresponding diameter to get a $(6,3)$-sequence.*

Proof. Since the sequence is non-alternating, there exist two points on one side of the diameter with no point on the other side whose image lies between them on the circumference. Rotate the diameter until it almost touches the first of such two points. Using the new diameter, we either get a $(6,3)$-sequence, or otherwise we get a $(5,4)$-sequence. For the latter case, we still rotate the diameter until it passes the two points (and no other points) to get a $(6,3)$-sequence. $\square$

What remains is to check the $(5,4)$ alternating sequences. Assume, without loss of generality, that such sequences start with R and contain the subsequence g, G, g. There are 22 such possibilities for $\mathcal{S}$, each with 3 (not only 2) of the claimed triangles. We enumerate these cases together with the corresponding triangles in Table 2.

Note that the sequences $6-10$ are obtained from the sequences $1-5$ by reversing each sequence, and swapping every R with B and every r with b. Also, the sequences $15-18$ are obtained by reversing each of the sequences $11-14$.

3 A 3-dimensional counter example

Theorem 1 cannot be generally extended to sets of 3-dimesnional points, as the following example illustrates. Let $v_1 = (1,0,0)$, $v_2 = (-1,0,0)$, $v_3 = (0,1,0)$, $v_4 = (0,-1,0)$, $v_5 = (0,0,1)$, $v_6 = (0,0,-1)$, $v_7 = (-1,-1,-1)$, $v_8 = (1,1,-1)$, $v_9 = (1,-1,1)$, $v_{10} = (-1,1,1)$. Let $S_1 = \{v_1, v_2\}$, $S_2 = \{v_3, v_4\}$, $S_3 = \{v_5, v_6\}$, $S_4 = \{v_7, v_8, v_9, v_{10}\}$, and $Z = (0,0,0)$ (see Figure 1). Then, clearly, $Z \in \operatorname{conv} S_i$, for $i = 1, \ldots, 4$. Furthermore, any two sets $T_1, T_2 \subset S_1 \cup S_2 \cup S_3 \cup S_4$, such that $Z \in \operatorname{conv} T_i$ and $|T_i \cap S_j| \leq 1$, for $i = 1, 2$ and $j = 1, 2, 3, 4$, must also have $T_1 \cap T_2 \neq \emptyset$. Indeed, consider any two such sets T_1 and T_2. It is easy to verify that $|T_i \cap S_j| = 1$, for all $i = 1, 2$ and $j = 1, 2, 3, 4$. Then, there are only four possibilities for each of T_1 and T_2, namely $\{v_1, v_3, v_5, v_7\}$, $\{v_4, v_5, v_8, v_9\}$, $\{v_1, v_3, v_6, v_9\}$, or $\{v_1, v_4, v_6, v_{10}\}$. But, no two of these four sets are disjoint.

It is worth mentioning that whether the stronger version of Bárány's Theorem holds or not can be verified, for any fixed dimension d, by a computer program, using similar ideas to those we gave for $d = 2$.

	$\mathcal{S}$	Tri. 1	Tri. 2	Tri. 3
1	$R, r, R, g, B, b, G, g, B$	R, g, B	r, B, g	R, b, G
2	$R, r, B, b, R, g, G, g, B$	R, g, B	r, B, g	R, b, G
3	$R, r, B, g, R, b, G, g, B$	R, g, B	r, B, g	R, b, G
4	$R, r, B, g, G, b, R, g, B$	R, g, B	r, B, g	G, b, R
5	$R, g, B, r, R, b, G, g, B$	R, g, B	g, B, r	R, b, G
6	$R, g, G, r, R, g, B, b, B$	R, g, B	g, R, b	G, r, B
7	$R, g, G, g, B, r, R, b, B$	R, g, B	g, R, b	G, r, B
8	$R, g, G, r, B, g, R, b, B$	R, g, B	g, R, b	G, r, B
9	$R, g, B, r, G, g, R, b, B$	R, g, B	g, R, b	B, r, G
10	$R, g, G, r, B, b, R, g, B$	R, g, B	b, R, g	G, r, B
11	$R, r, B, b, B, g, G, g, R$	R, b, G	r, B, g	B, g, R
12	$R, r, B, g, G, b, B, g, R$	R, g, B	r, B, g	G, b, R
13	$R, g, B, b, B, r, G, g, R$	R, b, G	g, B, r	B, g, R
14	$R, g, B, b, G, r, B, g, R$	R, g, B	b, G, r	B, g, R
15	$R, g, G, g, B, b, B, r, R$	G, b, R	g, B, r	R, g, B
16	$R, g, B, b, G, g, B, r, R$	B, g, R	g, B, r	R, b, G
17	$R, g, G, r, B, b, B, g, R$	G, b, R	r, B, g	R, g, B
18	$R, g, B, r, G, b, B, g, R$	B, g, R	r, G, b	R, g, B
19	$R, r, R, g, G, g, B, b, B$	R, g, B	r, G, b	R, g, B
20	$R, r, B, g, G, g, R, b, B$	R, g, B	r, G, b	B, g, R
21	$R, g, B, b, G, r, R, g, B$	R, g, B	b, G, r	R, g, B
22	$R, g, B, r, G, b, R, g, B$	R, g, B	r, G, b	R, g, B

Table 2: The sequence $\mathcal{S}$ and the corresponding three claimed triangles.

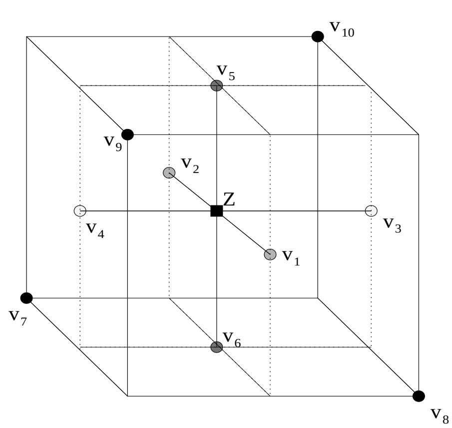

Figure 1: An example showing that Theorem 1 cannot be extended to 3 dimensions.

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