On the Number of Pseudo-Triangulations of Certain Point Sets

Oswin Aichholzer* David Orden† Francisco Santos† Bettina Speckmann‡

Abstract

We pose a monotonicity conjecture on the number of pseudo-triangulations of any planar point set, and check it in two prominent families of point sets, namely the so-called double circle and double chain. The latter has asymptotically $12^n n^{\Theta(1)}$ pointed pseudo-triangulations, which lies significantly above the maximum number of triangulations in a planar point set known so far.

1 Introduction

Pseudo-triangulations, also called geodesic triangulations, are a generalization of triangulations which has found multiple applications in Computational Geometry in the last years. They were originally studied in the context of visibility [10, 11] and ray shooting [5, 6], but recently have also been used in kinetic collision detection [1, 8], rigidity [17], and guarding [16].

A *pseudo-triangle* is a planar polygon that has exactly three convex vertices, called *corners*, with internal angles less than π. A *pseudo-triangulation* for a set A of n points in the plane is a partition of $\text{conv}(A)$ into pseudo-triangles whose vertex set is exactly A. A vertex is *pointed* if it has an adjacent angle greater than π.

The set of all pseudo-triangulations of a point set has somewhat nicer properties than that of all triangulations. For example, pseudo-triangulations of a point set with n elements form the vertex set of a certain polyhedron of dimension $3n-3$ [9]. The diameter of the graph of pseudo-triangulations is $O(n \log n)$ [3] versus the $\Theta(n^2)$ diameter of the graph of triangulations of certain point sets. For standard triangulations it is not known which sets of points have the fewest or the most triangulations, but it was shown in [2] that sets of points in convex position minimize the number of pointed pseudo-triangulations.

Let A be a point set and let A_I be its subset of interior points. Let $PT(A)$ be the set of pseudo-triangulations of A. For each subset $W \subseteq A_I$ we denote by $PT_W(A)$ the set of pseudo-triangulations of A in which the points of W are pointed and those of $A_I \setminus W$ are non-pointed. For example, $PT_\emptyset(A)$ and $PT_{A_I}(A)$ are the triangulations and the pointed pseudo-triangulations of A, respectively. In [12], the following inequality is proved:

$$3\,|PT_{W\setminus\{v\}}(A)| \geq |PT_W(A)|.$$

We pose the following conjecture in the opposite direction, implicit in previous work:

Conjecture 1 *For every point set A in general position in the plane, the cardinalities of $PT_W(A)$ are monotone with respect to W. That is to say, for any subset W of its interior points and for every $v \in W$, one has*

$$|PT_W(A)| \geq |PT_{W\setminus\{v\}}(A)|.$$

In this paper we consider three families of point sets in the plane: double circles, double chains and what we call single chains. The first two are the examples with asymptotically the smallest and biggest number of triangulations known (see, e.g., [4, 14]). The third is studied as a step to analyze double chains. Our goal is twofold: check Conjecture 1, and compare the numbers $|PT_{A_I}(A)|$ and $|PT(A)|$ to the number $|PT_\emptyset(A)|$ of triangulations in these point sets. Our results are summarized in the following table. In all cases the set has n points and a polynomial factor has been neglected.

	double circle	single chain	double chain
$\lvert PT_\emptyset(A)\rvert$	$\sqrt{12}^n$	4^n	8^n
$\lvert PT_{A_I}(A)\rvert$	$\sqrt{28}^n$	8^n	12^n
$\lvert PT(A)\rvert$	$\sqrt{40}^n$	12^n	?
Conjecture 1	YES	YES	?

Conjecture 1 would imply that the total number of pseudo-triangulations in the double chain is between 16^n and 24^n. We have reasons to believe it to be 20^n. We show that Conjecture 1 in the double chain would follow from a certain conjecture on pointed pseudo-triangulations of the single chain.

2 The double circle and its relatives

For any given pair of positive integers $v \geq 3$ and $i \leq v$, a point set in *almost convex position* with parameters (v, i) consists of a set of v points forming the vertex set of a convex v-gon and a set of i interior points, placed

*Institute for Software Technology, Graz University of Technology, oaich@ist.tugraz.at

†Department of Mathematics, Statistics and Computer Science, University of Cantabria, ordend@matesco.unican.es, santosf@unican.es

‡Department of Computing Science, TU Eindhoven, speckman @win.tue.nl

sufficiently close to i different edges of the v-gon. The *double circle* is the extremal case $v = i$.

This situation is a special case of what is called "almost-convex polygons" in [7]. There it is shown that the number of triangulations of such a point set does not depend on the choice of the i edges of the v-gon. Indeed, if we call this number $n(v, i)$, the recursive formula

$$n(v, i) = n(v + 1, i - 1) - n(v, i - 1).$$

allows to compute $n(v, i)$ starting with $n(v, 0) = C_{v-2}$ (Catalan numbers). The array obtained by this recursion (difference array of Catalan numbers) appears in Sloane's Online Encyclopedia of Integer sequences [15] with ID number A059346. The double circle, with $v = i = n/2$ has asymptotically $\Theta(\sqrt{12}^n n^{-3/2})$ triangulations [14]. It is conjectured in [4] that this is the smallest number of triangulations that n points in general position in the plane can have.

Let p be a specific interior point of a set A in almost convex position and let qr be the convex hull edge which has p next to it. Let B and C be the point sets obtained respectively by deleting p from A and by moving p to convex position across the edge qr (see Fig. 1).

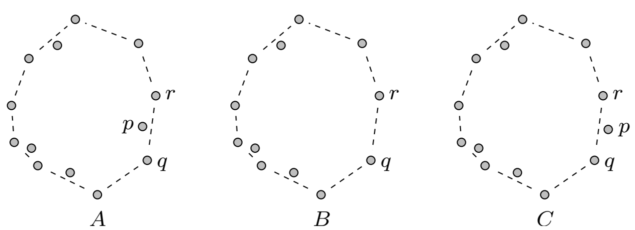


Figure 1: An almost convex point set A with $v = 9$ and $i = 4$.

Lemma 1 *For every $W \subset A_I$ not containing p (so that W is also a set of interior points of B and C) one has:*

1. $|PT_W(A)| = |PT_W(C)| - |PT_W(B)|$.
2. $|PT_{W\cup\{p\}}(A)| = 2\,|PT_W(C)| - |PT_W(B)|$.

Proof: 1. It is clear that there are bijections between: (1) pseudo-triangulations of C pointed at W that use the edge qr and pseudo-triangulations of B pointed at W and (2) pseudo-triangulations of C pointed at W that do not use the edge qr and pseudo-triangulations of A pointed at W. These bijections give the statement.

2. Pseudo-triangulations of A in which p is pointed come in three flavors: those using the edges pq and pr (and hence having no other edge incident to p), those using pq but not pr, and those using pr but not pq. The first set is in bijection with the pseudo-triangulations of B. Each of the other two is in bijection with pseudo-triangulations of C that do not use the edge qr, that is pseudo-triangulations of C minus those of B. Since the bijections preserve pointedness at interior points (other than p), we get $|PT_{W\cup\{p\}}(A)| = |PT_W(B)| + 2\,(|PT_W(C)| - |PT_W(B)|)$, as desired. □

Observe that Lemma 1 implies that A satisfies Conjecture 1 and that the number $|PT_W(A)|$ depends only on v, i and $k := |W|$. Let $n(v, i, k)$ denote this number. Since $n(v, i, 0) = 4^v 3^i$ (modulo a polynomial factor) and since

$$n(v, i, k) = 2(v + 1, i - 1, k - 1) - n(v, i - 1, k - 1),$$

we conclude that $n(v, i, k) \sim 4^v 3^{i-k} 7^k$, modulo a polynomial factor. Adding the numbers over all the possible subsets of interior points gives

$$\sum_{k=0}^{i} \binom{i}{k} 4^v 3^{i-k} 7^k = 4^v 10^i.$$

Hence, a double circle (the case $i = v = n/2$) has $\sqrt{28}^n$ pointed pseudo-triangulations and $\sqrt{40}^n$ pseudo-triangulations in total, modulo a polynomial factor.

3 The single chain

By a *single chain* we mean the following point set A: three extremal vertices and a concave chain of l points $p_1, \ldots, p_l$ next to an edge. Let p be the extremal point opposite to the chain. We call p the *top* of A.

We classify the pointed pseudo-triangulations of the single chain according to which interior points are joined to the top. For any subset $W \subset A_I$ we denote by $PPT_W(A)$ the set of pointed pseudo-triangulations of A in which p is joined to p_i if and only if $p_i \in W$. For example, $PPT_\emptyset(A)$ is in bijection to the set of triangulations of the convex $l + 2$-gon, hence its cardinality is the Catalan number C_l. It is easy to show that:

Lemma 2 *For every W:*

$$|PT_W(A)| = \sum_{W' \subset W} |PPT_{W'}(A)|.$$

Hence, Conjecture 1 holds for A.

For example, the equality $|PT_\emptyset(A)| = |PPT_\emptyset(A)|$ is the easy observation that triangulations of A are in bijection to triangulations of the convex $l + 2$-gon. Curiously enough, $PPT_{A_I}(A)$ (that is, the pointed pseudo-triangulations in which the top point p is joined to everything), have the cardinality of the next Catalan number C_{l+1}, and flips between them form the graph of the corresponding associahedron, by Section 5.3 of [13] (see also the remark and picture on pp. 728–729). The following is a 1-dimensional analog of Conjecture 1.

Conjecture 2 *For every $W \subset A_I$ and $p \in A_I \backslash W$,*

$$|PPT_{W\cup\{v\}}(A)| \geq |PPT_W(A)|.$$

We do not know how to compute the numbers $PPT_W(A)$. But we can compute the sum of all the $PPT_W(A)$'s for each cardinality of W, via the following recursive formulae whose bijective proof we omit.

Theorem 3 *Let* $a(l,i) := \sum_{|W|=i} |PPT_W(A)|$. *Then:*

1. $a(l,0) = C_l$, *and* $a(l,1) = (l+1)C_l$.
2. *For every* $i \geq 2$,

$$a(l,i) = \binom{l+1}{i} C_l - a(l-1, i-2).$$

The first few values of $a(l,i)$ are as follows:

$l \setminus i$	0	1	2	3	4	5	$a_l = \sum a(l,i)$
0	1						1
1	1	2					3
2	2	6	5				13
3	5	20	28	14			67
4	14	70	135	120	42		381
5	42	252	616	770	495	132	2307

The recursion also tells us that the array $a(l,i)$ equals the sequence A062991 in [15]. The row sums, that is, the numbers $|PT_{A_I}(A)|$ of pointed pseudo-triangulations, form the sequence A062992. We can obtain them adding over all values of i in the formula of Theorem 3:

Corollary 4 *The number* $a_l = |PT_{A_I}(A)|$ *of pointed pseudo-triangulations of the single chain satisfies:*

$$a_l = 2^{l+1} C_l - a_{l-1}.$$

Hence, $a_l = |PT_{A_I}(A)| \in \Theta(8^l l^{-3/2})$.

We now turn our attention to the total number of pseudo-triangulations $PT(A)$. Lemma 2 implies that:

$$|PT(A)| = \sum_{W' \subset A_I} 2^{|A_I - W'|} |PPT_{W'}(A)| = \sum_{i=0}^{l} 2^{l-i} a(l,i).$$

Corollary 5 *The number* $b_l = |PT(A)|$ *of pseudo-triangulations of the single chain satisfies:*

$$2b_l = 3^{l+1} C_l - b_{l-1}.$$

Hence, $b_l = |PT(A)| \in \Theta(12^l l^{-3/2})$.

4 The double chain

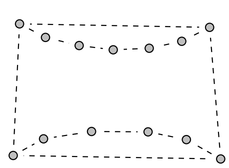


Figure 2: A double chain: $l = 5$ and $m = 4$.

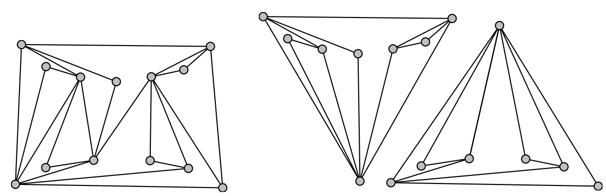
For any two numbers $l, m \geq 0$, a double chain is a convex 4-gon with l and m points, respectively, placed forming concave chains next to opposite edges of the 4-gon in a way that the they do not cross the two diagonals of the convex 4-gon (see Fig. 2). The double chain has exactly

$$C_l C_m \binom{l+m+2}{l+1}$$

triangulations. In the extremal case $l = m = (n-4)/2$ this gives $\Theta(8^n n^{-7/2})$. This is (asymptotically) the point set with the largest number of triangulations known so far.

Let A be a double chain with l and m interior points in the two chains, resp. (so A has $l+m+4$ points in total). We call the $l+2$ and $m+2$ vertices in the two chains the "top" and "bottom" parts.

To count the number of pointed pseudo-triangulations of A, let us call B and C single chains with l and m interior points each. B can be considered the subset of A consisting of the top part plus a bottom vertex, and analogously for C. Every pseudo-triangulation T_A of A induces pseudo-triangulations T_B and T_C of B and C as follows: consider on the one hand all the pseudo-triangles of T_A that use at most one vertex of the bottom, and contract these vertices to a single one. Do the same for pseudo-triangles with at most one vertex in the top (see Fig. 3).



(a) T_A (b) T_B and T_C

Figure 3: Decomposing a pseudo-triangulation of a double chain.

Conversely, given a pair of pseudo-triangulations of B and C, if i (resp. j) denotes the number of interior edges incident to the bottom (resp. top) point, there are exactly $\binom{i+j+2}{i+1}$ ways to recover a pseudo-triangulation of A from that data, by shuffling the $i+1$ pseudo-triangles of T_B incident too the bottom and the $j+1$ of T_C incident to the top.

Theorem 6 *Let* V *and* W *be subsets of the top and bottom interior points. For each* $V' \subset V$ *and* $W' \subset W$ *let* $t^{V,W}_{V',W'} = \binom{l-|V \setminus V'|+m-|W \setminus W'|+2}{l-|V \setminus V'|+1}$. *Then:*

$$|PT_{V \cup W}(A)| = \sum_{\substack{V' \subset V \\ W' \subset W}} t^{V,W}_{V',W'} |PPT_{V'}(B)||PPT_{W'}(C)|.$$

Proof: The first observation is that the "shuffling" described above preserves pointedness. The second observation is that in the expression

$$|PT_V(B)| = \sum_{V' \subset V} |PPT_{V'}(B)|$$

of Lemma 2, each element of $PPT_{V'}(B)$ corresponds to an element of $PT_V(B)$ with exactly $l - |V \setminus V'|$ interior edges incident to the bottom point (same for C). □

Corollary 7 *If Conjecture 2 holds, then the double chain satisfies Conjecture 1.*

We now consider the number of pointed pseudo-triangulations of the double chain. Theorem 6 can be rewritten in this case as:

$$|PT_{A_I}(A)| = \sum_{i=0}^{l}\sum_{j=0}^{m}\binom{i+j+2}{i+1}a(l,i)a(m,j),$$

where $a(\cdot,\cdot)$ is as in the previous section. The following triangular array gives these numbers for $l+m \leq 5$.

2					
8	8				
42	38	42			
252	226	226	252		
1630	1502	1476	1502	1630	
11048	10618	10604	10604	10618	11048

The sequence for $l = m$ is

$$2, 38, 1476, 81310, 5495276, 424398044, \ldots$$

In order to analyze the asymptotics of this sequence we need the following lemma on the numbers $a(l,i)$:

Lemma 8

$$1 - \frac{i(i-1)}{(4l-2)(l-i+2)} \leq \frac{a(l,i)}{\binom{l+1}{i}C_l} \leq 1$$

Corollary 9 *Let A be a double chain with n points and with equal numbers on both sides (that is to say, $l = m = (n-4)/2$). Then:*

$$\Omega(12^n n^{-9/2}) \leq |PT_{A_I}(A)| \leq O(12^n n^{-3/2}).$$

Acknowledgments

Parts of this work were done while the authors visited the Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya. Research of Oswin Aichholzer is partially supported by Acciones Integradas 2003-2 004, Proj.Nr.1/2003. Research of Francisco Santos and David Orden is partially supported by Acción Integrada España-Austria HU2002-0010 and grant BFM2001-1123 of Spanish Dirección General de Investigación Científica.

References

[1] P. K. Agarwal, J. Basch, L. J. Guibas, J. Hershberger, and L. Zhang. Deformable free space tilings for kinetic collision detection. In *Proc. 5th Workshop Algorithmic Found. Robotics)*, pages 83–96. A. K. Peters, 2001.

[2] O. Aichholzer, F. Aurenhammer, H. Krasser, and B. Speckmann. Convexity Minimizes Pseudo-Triangulations. In *Proc. 14th Canad. Conf. Comp. Geom.*, pages 158–161, 2002.

[3] O. Aichholzer, F. Aurenhammer, and H. Krasser. Adapting (pseudo)-triangulations with a near-linear number of edge flips. In *Proc. 8th Intern. Workshop Alg. and Data Struct.*, 2003. (to appear).

[4] O. Aichholzer, H. Krasser. The point-set order-type database: A collection of applications and results. In *Proc. 13th Canad. Conf. Comp. Geom.*, pages 17–20, 2001.

[5] B. Chazelle, H. Edelsbrunner, M. Grigni, L. J. Guibas, J. Hershberger, M. Sharir, and J. Snoeyink. Ray shooting in polygons using geodesic triangulations. *Algorithmica*, 12:54–68, 1994.

[6] M. Goodrich and R. Tamassia. Dynamic ray shooting and shortest paths in planar subdivision via balanced geodesic triangulations. *J. Algorithms*, 23:51–73, 1997.

[7] F. Hurtado and M. Noy. Counting triangulations of almost-convex polygons. *Ars Combinatoria*, 45:169–179, 1997.

[8] D. Kirkpatrick, J. Snoeyink, and B. Speckmann. Kinetic collision detection for simple polygons. *Intern. Journal Comp. Geom. Appl.*, 12:3–27, 2002.

[9] D. Orden and F. Santos. The polytope of non-crossing graphs on a planar point set. Preprint 2003, http://arxiv.org/abs/math.CO/0302126.

[10] M. Pocchiola and G. Vegter. Minimal tangent visibility graphs. *Comput. Geom. Theory Appl.*, 6:303–314, 1996.

[11] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. *Discrete Comp. Geom.*, 16:419–453, 1996.

[12] D. Randall, G. Rote, F. Santos and J. Snoeyink. Counting triangulations and pseudo-triangulations of wheels. In *Proc. 13th Canad. Conf. Comput. Geom.*, pages 149-152, 2001.

[13] G. Rote, F. Santos, and I. Streinu. Expansive motions and the polytope of pointed pseudo-triangulations. *Discrete and Computational Geometry – The Goodman-Pollack Festschrift*, (B. Aronov, S. Basu, J. Pach, M. Sharir, eds), Algorithms and Combinatorics, Springer Verlag, Berlin, pages 699-736, 2003.

[14] F. Santos, R. Seidel, A better upper bound on the number of triangulations of planar point sets. *J. Combin. Theory Ser. A*, 102:186–193, 2003.

[15] Neil J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. Copyright 2003 AT&T, http://www.research.att.com/ njas/sequences.

[16] B. Speckmann and C. D. Tóth. Allocating vertex π-guards in simple polygons via pseudo-triangulations. In *Proc. 14th Symp. Disc. Alg.*, pages 109–118, 2003.

[17] I. Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *Proc. 41st FOCS*, pages 443–453, 2000.