A Constant-Factor Approximation for Maximum Weight Triangulation

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Abstract

The paper is the first report on approximation algorithms for computing the maximum weight triangulation of a set of n points in the plane. We prove an $\Omega(\sqrt{n})$ lower bound on the approximation factor for several heuristics: maximum greedy triangulation, maximum greedy spanning tree triangulation and maximum spanning tree triangulation. We then propose the Spoke Triangulation algorithm, which always approximates the maximum weight triangulation for points in general position within a factor of six and can be computed in $O(n \log n)$ time. We also prove that Spoke Triangulation approximates the maximum weight triangulation of a convex polygon within a factor of two.

1 Introduction

Let P be a set of n points in the plane. A *triangulation* $T(P)$ of P is a maximal set of non-intersecting straight-line segments connecting points in P. The *weight* $|T(P)|$ of $T(P)$ is the sum of the Euclidean lengths of edges in $T(P)$. The *minimum weight triangulation* (MWT) is a triangulation of P with minimum weight. The *maximum weight triangulation* (MAT) is a triangulation of P with maximum weight. Computing the former is a well known problem in computational geometry and neither known to be NP-complete, nor known to be solvable in polynomial time [1].

The best approximation result to MWT(P) is due to Levcopoulos and Krznaric [5]. Their algorithm produces a (very large) constant factor approximation to the MWT(P). Other known heuristic algorithms were described in [7], [6] and [3]. Lingas [6] and Heath and Pemmaraju [3] introduced so-called *minimum spanning tree triangulation* and *greedy spanning tree triangulation* heuristics. Levcopoulos and Krznaric [4] showed that these heuristics have approximation ratio $\Omega(n)$, $\Omega(\sqrt{n})$, respectively, in the worst case.

In contrast to the extensive literature on minimum weight triangulation, there is only one published result [9] on maximum weight triangulation to the best of the author's knowledge. In that paper, the authors gave an $O(n^2)$ algorithm for finding the MAT of an n-sided polygon inscribed in a disk.

In this paper, we prove that the worst-case approximation lower bound for MAT by maximum greedy triangulation, maximum greedy spanning tree triangulation and maximum spanning tree triangulation heuristics (which are the natural analogues of their MWT counterparts) is $\Omega(\sqrt{n})$. We also obtain a tight bound of $\Theta(n)$ for degenerate cases. Our construction is more complicated than its counterpart for minimum weight triangulation. We then propose the *Spoke Triangulation* and prove that it approximates the maximum weight triangulation for points in general position within a factor of 6 and can be computed in $O(n \log n)$ time. We also prove that Spoke Triangulation approximates the maximum weight triangulation of a convex polygon within a factor of 2.

We begin with an easy observation and some definitions. Gilbert [2] showed that the minimum weight triangulation of a simple polygon can be computed in $O(n^3)$ time by dynamic programming. In an analogus manner, its maximum weight triangulation can also be computed in $O(n^3)$ time. The *greedy triangulation* GT(P) of P is obtained by repeatedly adding a longest possible edge that does not properly intersect any of the previously generated edges. The *greedy spanning tree triangulation* GSTT(P) of P is obtained as follows: Start with a maximum spanning tree of the greedy triangulation, triangulate optimally each of the simple polygons bounded by this spanning tree and the convex hull of P. The *maximum spanning tree triangulation* MSTT(P) of P is obtained similarly to GSTT(P) except that it starts with the Euclidean maximum non-crossing spanning tree of P.

2 Lower bounds for GT, GSTT and MSTT

Theorem 1. *For any integer $n \geq 0$, there exists a set P of n points such that $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GT}(P)|} = \Omega(\sqrt{n})$, $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GSTT}(P)|} = \Omega(\sqrt{n})$, and $\frac{|\mathrm{MAT}(P)|}{|\mathrm{MSTT}(P)|} = \Omega(\sqrt{n})$.*

Proof. We first construct a point set P for which $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GT}(P)|} = \Omega(\sqrt{n})$. We start by allowing degeneracies, that is, we permit three or more collinear points in P. Assume the points in P are distributed as shown in Figure 1(a). All the points are on the sides of the

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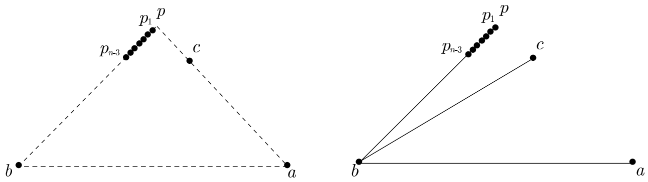


Figure 1: (a) Degenerate case (b) Maximum spanning tree

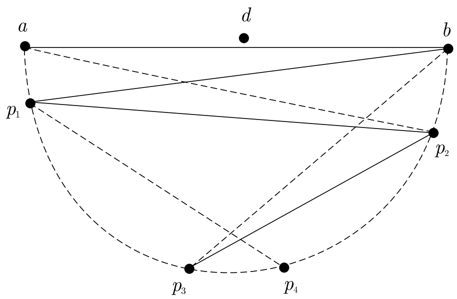


Figure 2: Non-degenerate case

right triangle apb, where p is only an imaginary point ($p \notin P$) satisfying $\angle apb = \pi/2, |ab| = n, |pb| = |pa| = n/\sqrt{2}, |pc| = 1, |pp_j| < 1$ for $1 \le j \le n-3$.

By the definition of greedy triangulation, ab, bc, bp_{n-3} (recall that $p \notin P$) will be the first three edges to be added. After that, there is only one possible way to complete the triangulation: to connect p_j to c for $1 \le j \le n-3$, connect p_j to p_{j+1} for $1 \le j \le n-4$ and connect a to c. Note that $|ab|+|bc|+|bp_{n-3}|+|ac| < 4n$ and $|p_jc| < \sqrt{2}, 1 \le j \le n-3$, so $|\mathrm{GT}| = |ab| + |bc| + |bp_{n-3}| + \sum_{j=1}^{n-4} |p_ip_{i+1}| + \sum_{j=1}^{n-3} |p_ic| + |ac| = \Theta(n)$. On the other hand, a larger weight triangulation $LWT(P)$ (clearly $|LWT(P)| \le |\mathrm{MAT}(P)|$) is obtained by first connecting p_j to a for $1 \le j \le n-3$. Then $|LWT| \ge \sum_{j=1}^{n-3} |p_ja| = \Theta(n^2)$, thus $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GT}(P)|} = \Omega(n)$. This bound is tight immediately from Observation 3.

Next we consider the non-degenerate case, that is, we do not allow three or more collinear points. In [4], the authors started with the degenerate construction and then just moved the points p_j ($1 \le j \le n-3$) from a line to a curve (parabola). However, it is not hard to see this will not work in our case. Our idea is roughly that we distribute the first $\sqrt{n}$ points in a nice way on a unit half-circle such that the total length of the first $\sqrt{n}$ edges in $\mathrm{GT}(P)$ is $O(\sqrt{n})$. Then we claim that the total length of all remaining edges is also $O(\sqrt{n})$. However $\mathrm{MAT}(P)$ has weight $\Omega(n)$, which will complete the proof.

Refer to Figure 2. Let ab be the diameter of the unit half-circle C and let d lie slightly above the center of C. So $|\widehat{ab}| = \pi$. We place all other points inside or on C. We say p is *below* a chord xy when p lies in or on the boundary of the closed region bounded by arc $\widehat{xy}$ and chord xy, then $|xy| > |xp|, |yp|$ (note that p lies in the closed half-disk). We distribute the next $k-2$ points denoted by $p_1, p_2, \ldots, p_{k-2}$ on C as follows.

Place p_1 on C such that $|\widehat{bp_1}| = \pi/x_1$, where $1 < x_1 < 2$. We place remaining points below bp_1. We want to add bp_1 to $\mathrm{GT}(P)$ and we can do this if all remaining points lie below the chord p_1p_2 such that $|\widehat{bp_1}| > |\widehat{ap_2}|$. Let $x_2 = \frac{|\widehat{p_1b}|}{|\widehat{p_1p_2}|} > 1$ and restrict $x_2 < 2$. Then $x_2 > \frac{1}{2-x_1}$. Next we will add p_1p_2 to $\mathrm{GT}(P)$ if all other points lie below the chord p_2p_3 such that $|\widehat{p_1p_2}| > |\widehat{bp_3}|$. Let $x_3 = \frac{|\widehat{p_2p_1}|}{|\widehat{p_2p_3}|} > 1$ and restrict $x_3 < 2$. Then $x_3 > \frac{1}{2-x_2}$.

Generally we place the points incrementally. Suppose we have just added the edge $p_{j-1}p_j$, that is, the last point we placed is p_{j+1}, and all remaining points lie below p_jp_{j+1} and we have determined $x_1, x_2, \ldots, x_{j+1}$ where $x_{j+1} = \frac{|\widehat{p_jp_{j-1}}|}{|\widehat{p_jp_{j+1}}|}$ and thus $|\widehat{p_{j-1}p_{j+1}}| = (1 - \frac{1}{x_{j+1}})|\widehat{p_jp_{j-1}}|, |\widehat{p_jp_{j+1}}| = \pi/\prod_{k=1}^{j+1} x_k$.

Now we want to add edge p_jp_{j+1}, thus we should place the point p_{j+2} on C such that all the remaining points lie below $p_{j+1}p_{j+2}$ guaranteeing $|\widehat{p_jp_{j+1}}| > |\widehat{p_{j-1}p_{j+2}}|$. Let $x_{j+2} = \frac{|\widehat{p_{j+1}p_j}|}{|\widehat{p_{j+1}p_{j+2}}|}$, then $|\widehat{p_{j+1}p_{j+2}}| = |\widehat{p_jp_{j+1}}|/x_{j+2} = \pi/\prod_{k=1}^{j+2} x_k$. It is easy to see $|\widehat{p_jp_{j+1}}| > |\widehat{p_{j-1}p_{j+1}}| + |\widehat{p_{j+1}p_{j+2}}|$ and $(1 - \frac{1}{x_{j+2}})|\widehat{p_jp_{j+1}}| > (1 - \frac{1}{x_{j+1}})|\widehat{p_jp_{j-1}}|$. Thus $x_{j+2} > \frac{1}{2-x_{j+1}}$.

Let $x_j = 1/(2-x_{j-1}) + \epsilon$ ($\epsilon > 0$). Suppose we have added the points $p_1, p_2, \ldots, p_{k-2}$ and determined $x_1, x_2, \ldots, x_{k-2}$. If $1 < x_j < 2$ for $j = 1, 2, \ldots, k-2$, then $x_1, x_2, \ldots, x_{k-2}$ is a valid sequence for our example, i.e., $x_j > 1/(2-x_{j-1})$. Then it is not hard to see that $1/(2-x_{j-1}) > x_{j-1}$, thus $x_1, x_2, \ldots, x_{k-2}$ is a monotonic increasing sequence. This is why we did not place the points symmetrically on the circle C. Lemma 2 guarantees the condition if select $x_1 = (k+1)/k$, $\epsilon = 1/k^k$, and $x_j = \frac{1}{2-x_{j-1}} + \frac{1}{k^k}$, for $j = 2, 3, \ldots, k-2$. Since any chord in C is shorter than its corresponding arc, we can safely use the length of the arc to bound above the length of chord. The total length of the first $k-2$ edges is (note that $1/\prod_{j=1}^{i} x_j < 1/x_1^i$ since $x_1, x_2, \ldots, x_j$ is a monotonic increasing sequence) $|ab|+|bp_1|+\sum_{i=1}^{k-4} |p_ip_{i+1}| < \pi+\pi\sum_{i=1}^{k-4}(1/\prod_{j=1}^{i} x_j) < \pi+\pi\sum_{i=1}^{k-4}(1/x_1^i) < \pi + \frac{\pi}{x_1-1} = O(\frac{1}{x_1-1}) = O(k)$.

According to Lemma 2, $\prod_{i=1}^{k-3} x_i > \frac{k+1}{k}\frac{k}{k-1}\cdots\frac{5}{4} = \frac{k+1}{4}$, then $|\widehat{p_{k-4}p_{k-3}}| = \pi/\prod_{i=1}^{k-3} x_i = O(1/k)$. We now place the remaining $n-k-1$ points. Since they are distributed below $p_{k-3}p_{k-2}$, we restrict them to be arbitrarily distributed in an obtuse triangle $p_{k-3}p_{k-2}p_{k-1}$ where $p_{k-3}p_{k-2}$ is the longest edge. We also require p_id for $k \le i \le n-3$ must intersect $p_{k-4}p_{k-3}$. Since $p_{k-4}p_{k-3}$ has already been added to $\mathrm{GT}(P)$, no point below $p_{k-4}p_{k-3}$ can be visible to any point above it.

And since any edge we can add is bounded above by $|\widehat{p_{k-4}p_{k-3}}| = O(1/k)$, the length of any triangulation of the region bounded by $p_{k-4}p_{k-3}p_{k-2}p_{k-1}$ is bounded above by $O(1/k) \times \Theta(n-k) = O(n/k)$. In order to complete the triangulation, we also need to add the edges on convex hull whose length is bounded by π, then $|\mathrm{GT}| = O(k) + O(n/k) + \pi$. We select $k = \sqrt{n}$, then $|\mathrm{GT}| = O(\sqrt{n})$.

A larger weight triangulation $LWT(P)$ is easier to compute. We connect d to $p_1, p_2, \ldots, p_{k-2}$ and all other $n-k-1$ points $(p_{k-1}, p_{k+2}, \ldots, p_{n-3})$. Each of first $k-2$ edges is longer than $1/2$ and we have the following observation on the remaining $n-k-1$ edges. Through d we make a line perpendicular to $p_{k-3}p_{k-2}$ intersecting $p_{k-3}p_{k-2}$ at f. We can do so because d is very close to center of C. (In fact, even if they cannot intersect, the bound still holds.) Since $|p_{k-3}p_{k-2}| = O(1/k)$, then $|df| > \sqrt{1-1/k^2}$. Then each of the $n-k-1$ remaining edges is bounded below by $\sqrt{1-1/k^2}$. Thus (even if LWT is incomplete) $|LWT| > (k-2)/2 + (n-k-1)\sqrt{1-1/k^2} = \Omega(n)$ for $k = \sqrt{n}$. Then $|\mathrm{MAT}|/|\mathrm{GT}| = \Omega(\sqrt{n})$.

Finally we construct examples where $\frac{|\mathrm{MAT}(P)|}{|\mathrm{GSTT}(P)|} = \Omega(\sqrt{n})$, $\frac{|\mathrm{MAT}(P)|}{|\mathrm{MSTT}(P)|} = \Omega(\sqrt{n})$.

With degeneracies allowed, we reuse the example of Figure 1(a). The maximum spanning tree of $\mathrm{GT}(P)$ and maximum spanning tree of P are the same, see Figure 1(b). Then $|\mathrm{GSTT}| = |\mathrm{GT}|, |\mathrm{MSTT}| = |\mathrm{GT}|$ even if optimal algorithm is then used for completing the triangulation.

In non-degenerate case, we reuse the example in Figure 2. It is not hard to see that edges ab, bp_1, p_jp_{j+1} for $j = 1, 2, \ldots, k-4$ also belong to $\mathrm{MSTT}(P)$ and $\mathrm{GSTT}(P)$, thus $|\mathrm{GSTT}(P)|, |\mathrm{MSTT}(P)|$ are asymptotically the same as $|\mathrm{GT}(P)|$ since no point below $p_{k-4}p_{k-3}$ can be visible to any point above $p_{k-4}p_{k-3}$. Thus the approximation lower bound will not change. This completes the proof. □

Lemma 2. *For sufficiently large k and $\epsilon = \frac{1}{k^k}$, if a sequence $x_1, x_2, \ldots, x_{k-2}$ is defined as $x_1 = (k+1)/k$, and $x_j = 1/(2 - x_{j-1}) + \epsilon$, for $j = 2, 3, \ldots, k-2$, then*

$$1 < \frac{k-(j-2)}{k-(j-1)} < x_j < \frac{k-(j-1)}{k-j-\epsilon k^j} + \epsilon$$

for $j = 2, 3, \ldots, k-2$.

Proof. Omitted. □

3 A Constant-Factor Approximation for MAT

The *diameter* of P is the longest segment with both endpoints in P. For points in general position, we present a new triangulation algorithm, which approximates $\mathrm{MAT}(P)$ within a small constant factor. We call it the *Spoke Triangulation* algorithm (ST in short). Refer to Figure 3 and Algorithm 1.

Since any triangulation has at most $3n-6$ edges, the following observation is obvious

Observation 3. *For any triangulation $T(P)$ of P with a diameter D, $|T(P)| \leq (3n-6)D$.*

Theorem 4. *Spoke Triangulation algorithm properly triangulates a set of points P (in general position) in $O(n \log n)$ time and $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)| < 6$.*

Proof. It is obvious that the algorithm properly triangulates P. Now we bound the running time of $\mathrm{ST}(P)$. Step 1 requires $O(n \log n)$ time. Step 2 requires $O(n)$ time. Running time of step 3 is bounded above by sorting $p_1, p_2, \ldots, p_{n_1}$ which takes $O(n \log n)$ time. So the running time of $\mathrm{ST}(P)$ is $O(n \log n)$.

Finally we bound the approximation ratio. $\mathrm{ST}(P)$ is composed of diameter, p_ia, q_ib and the edges added in step 3. Suppose there exists $p_t \in R$ such that $|p_ta| < D/2$ which means $|p_tb| > D/2$. (For example, p'_2 in Figure 3.) Then according to ST, in the final triangulation there always exists a path from p_t to b containing no edge in $\{p_ia, q_ib, ab\}$. (For example, the path $p'_2p'_3b$ in Figure 3.) The length of the path is obviously longer than $|p_tb|$ which means $|ST| > D/2 + D + \sum_{i=1}^{n_1} |p_ia| + \sum_{i=1}^{n_2} |q_ib| > 1.5D + (n_1 + n_2)D/2 = nD/2 + D/2$.

Otherwise, there is no such point, i.e., for all $p_i \in R, |p_ia| > D/2$. According to ST, there must exist an empty triangle abp_k in the final triangulation (For example, abp'_5 in Figure 3.), then $|ST| > D + |p_ka| + |p_kb| + \sum_{p_i \in R - \{p_k\}} |p_ia| + \sum_{i=1}^{n_2} |q_ib| > nD/2 + D/2$.

In either case, by Observation 3 we have $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)| < (3n-6)D/(nD/2 + D/2) < 6$. □

Observation 5. *For convex polygon P, $|\mathrm{MAT}(P)| < nD + 0.15D$.*

Proof. We have n edges on the convex hull and $n-3$ edges in its interior. The perimeter of the convex hull is bounded above by πD according to [8]. Then $|\mathrm{MAT}(P)| < (n-3)D + \pi D < nD + 0.15D$. □

Since $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)| < (nD + 0.15D)/(nD/2 + D/2) < 2$, $\mathrm{ST}(P)$ approximates $\mathrm{MAT}(P)$ of convex polygon P within a factor of two and can be computed in $O(n \log n)$ time compared with $O(n^3)$ time needed to compute MAT exactly by using dynamic programming. Thus we have shown

Lemma 6. *If P is a set of n points in convex position, $\mathrm{ST}(P)$ properly triangulates P in $O(n \log n)$ time and $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)| < 2$.*

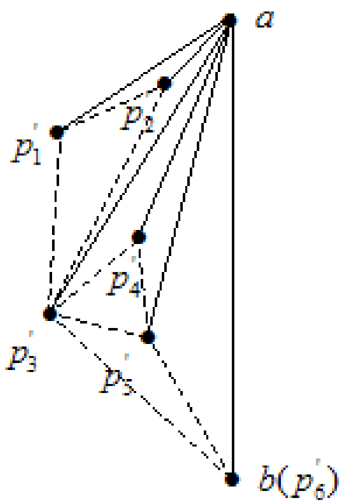
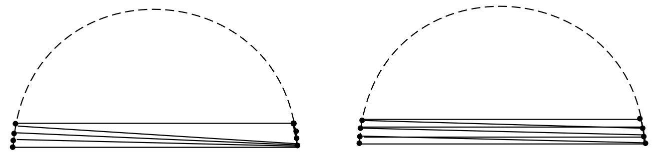


Figure 3: Spoke Triangulation (subset R)

Algorithm 1 Spoke Triangulation

1: Compute the diameter ab of P. Put $D = |ab|$. Let $R = \{p_i \mid p_i$ lies to the left of the line $ab\}$ and $S = \{q_i \mid q_i$ lies to the right of the line $ab\}$. Let n_1 (resp. n_2) denote the cardinality of R (resp. S). Then $n_1 + n_2 = n - 2$. Since at most one of R and S can be empty, without loss of generality, we assume R is not empty. Connect a to b.
2: At least one of $\sum_{i=1}^{n_1} |p_i a|$ and $\sum_{i=1}^{n_1} |p_i b|$ is larger than $n_1 D/2$ since $\sum_{i=1}^{n_1} |p_i a| + \sum_{i=1}^{n_1} |p_i b| > n_1 |ab|$ by triangle inequality. Without loss of generality we assume $\sum_{i=1}^{n_1} |p_i a| > n_1 D/2$ and connect all point p_i to a. Similarly, we connect all points q_i to b assuming $\sum_{i=1}^{n_2} |q_i b| > n_2 D/2$.
3: Now we have a spanning tree of P and we complete the triangulation as in Graham's scan algorithm for computing convex hull. And algorithm for triangulating the region to the left of ab is given below in lines 4–11. We triangulate the region to the right of it similarly.
4: Sort $p_1, p_2, \ldots, p_{n_1}$ according to the angle $p_i ab$ (from the largest to smallest), resulting in a sequence $p_1', p_2', \ldots, p_{n_1}'$. Let p_{n+1}' denote b.
5: Put the point p_1' in the list L_{left}.
6: **for** $i = 2$ **to** $n_1 + 1$ **do**
7:     Append p_i' to L_{left} and link p_i' to p_{i-1}'.
8:     **while** L_{left} contains more than two points **and** the last three points $(p_j', p_k', p_l', l < k < j)$ in L_{left} make a left turn (i.e., $\angle p_j' p_k' p_l' > \pi$ in quadrilateral $p_j' p_k' p_l' a$). **do**
9:         Connect p_j' to p_l' and remove p_k' from the list.
10:     **end while**
11: **end for**



(a) ST(P)   (b) MAT(P)

Figure 4: Lower bound for Lemma 6

Note that the above bound on approximation ratio is sharp since the example shown in Figure 4 indicates that $|\mathrm{MAT}(P)|/|\mathrm{ST}(P)|$ can be arbitrarily close to two. In the example, every point lies on a circle and two of them are the endpoints of the diameter. Half of the remaining points lie near to one endpoint of the diameter, and the other half near its other endpoint. We omit in this version an variant of the spoke triangulation algorithm that produces spanning tree of the same or larger weight by combining the spanning tree in ST with a greedy heuristic. However, we cannot guarantee that the weight of the entire triangulation will increase. This variant, however, triangulates correctly some examples (such as those in Figure 4(b)) that are "bad" for the spoke triangulation algorithm.

4 Conclusions

This paper is the first report of approximation algorithms for maximum weight triangulation to the best of the author's knowledge. We first prove that in the worst case maximum greedy triangulation, maximum greedy spanning tree triangulation and maximum spanning tree triangulation heuristics do not provide a constant factor approximation for the maximum weight triangulation. We then propose the Spoke Triangulation whose length is always within a small constant factor from the maximum. However, it is still a challenging problem to design a polynomial-time approximation scheme for MAT(P) of a general planar point set P. In fact, finding an algorithm that would guarantee a smaller constant factor approximation is interesting as well.

Acknowledgments

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