Transversals to line segments in $\mathbb{R}^3$

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Abstract

We completely describe the structure of the connected components of transversals to a collection of n arbitrary line segments in $\mathbb{R}^3$. We show that $n \geqslant 3$ line segments in $\mathbb{R}^3$ admit $0, 1, \ldots, n$ or infinitely many line transversals. In the latter case, the transversals form up to n connected components.

1 Introduction

Understanding the properties of lines in $\mathbb{R}^3$ is fundamental in computational geometry and graphics. A transversal to a set of line segments is a line that intersects all of them; such transversals have been of interest for years; see for instance [1].

Since a line in $\mathbb{R}^3$ has four degrees of freedom it can intersect at most four lines or line segments in general position. We address the following basic question. What is the possible geometry of the set of transversals to a collection of 4 arbitrary line segments? It is well-known that 4 lines in $\mathbb{R}^3$ admit at most 2 or infinitely many transversals [2, p. 164]. We prove that 4 arbitrary line segments admit up to 4 or infinitely many transversals. More generally, we determine the number of connected components of transversals to n line segments.

A line segment is either closed or open and may degenerate to a point. Two transversals to a collection of line segments are in the same *connected component* if and only if one can be continuously moved into the other while remaining a transversal to the collection of line segments. Equivalently, the two points in line space (e.g., in Plücker space) corresponding to the two transversals are in the same connected component of the set of points corresponding to all the transversals to the collection of line segments.

Our main result is the following theorem.

Theorem 1 *A collection of $n \geqslant 3$ arbitrary line segments in $\mathbb{R}^3$ admits $0, 1, \ldots, n$ or infinitely many transversals. In the latter case, the transversals can form any number, from 1 up to n inclusive, of connected components.*

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More precisely we show that, when $n \geqslant 4$, there can be more than 2 transversals only if the segments are in some degenerate configuration, namely if the n segments are members of one ruling of a hyperbolic paraboloid or a hyperboloid of one sheet, or if they are concurrent, or if they all lie in a plane with the possible exception of a group of one or more segments that all meet that plane at the same point.

Moreover in these degenerate configurations the number of connected components of transversals is as follows. If the segments are members of one ruling of a hyperbolic paraboloid, or if they are concurrent, their transversals form at most one connected component. If they are members of one ruling of a hyperboloid of one sheet, or if they are coplanar, their transversals can have up to n connected components (see Figures 1 and 6). Finally, if the segments all lie in a plane with the exception of a group of one or more segments that all meet that plane at the same point, their transversals can form up to $n-1$ connected components (see Figures 4 and 5).

A connected component of transversals may be an isolated line. For example, three segments forming a triangle and a fourth segment intersecting the interior of the triangle in one point have exactly three transversals (Figure 4 shows a similar example with infinitely many transversals). Also, the four segments in Figure 1 can be shortened so that the four connected components of transversals reduce to four isolated transversals.

An $O(n \log n)$-time algorithm for computing the transversals to n segments directly follows from the proof of Theorem 1. Indeed, we prove that the problem reduces to intersecting n intervals on a line or on a circle, or to computing the transversals to segments in $\mathbb{R}^2$ [1].

2 Proof of Theorem 1

Every non-degenerate line segment is contained in its *supporting line*. We define the supporting line of a point to be the vertical line through that point. We prove Theorem 1 by considering the three following cases which cover all possibilities but are not exclusive.

1. Three supporting lines are pairwise skew.

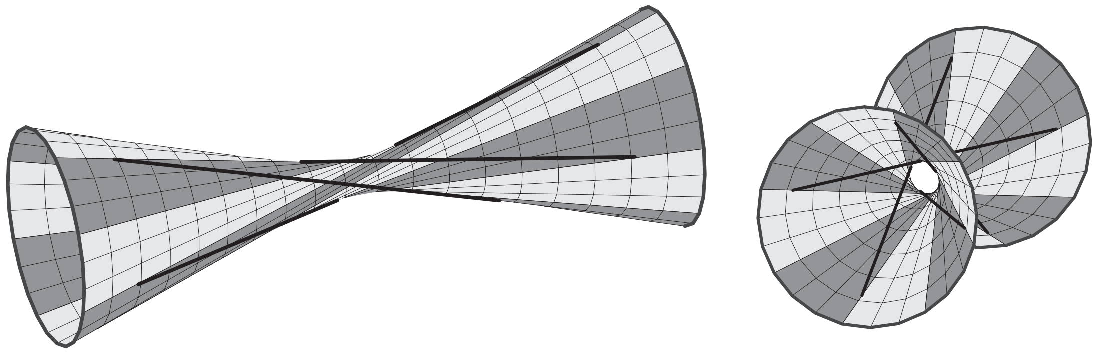

Figure 1: Two views of a hyperboloid of one sheet containing four line segments and their four connected components of transversals (corresponding to the shaded regions). The four segments are symmetric under rotation about the axis of the hyperboloid.

2. Two supporting lines are coplanar.

3. All the segments are coplanar.

We can assume in what follows that *the supporting lines are pairwise distinct.* Indeed, if disjoint segments have the same supporting line ℓ, then ℓ is the only transversal to those segments, and so the set of transversals is either empty or consists of ℓ. If non-disjoint segments have the same supporting line, then any transversal must meet the intersection of the segments. We can replace these overlapping segments by their common intersection.

2.1 Three supporting lines are skew

Three pairwise skew lines lie on a unique doubly-ruled hyperboloid, namely, a hyperbolic paraboloid or a hyperboloid of one sheet (see the discussion in [3, §3]). Furthermore, they are members of one ruling, say the "first" ruling, and their transversals are the lines in the "second" ruling that are not parallel to any of the three given skew lines.

Consider first the case where there exists a fourth segment whose supporting line ℓ does not lie in the first ruling. Either ℓ is not contained in the hyperboloid or it lies in the second ruling. In both cases, there are at most two transversals to the four supporting lines, which are lines of the second ruling that meet or coincide with ℓ (see Figure 2) [2, p. 164]. Thus there are at most two transversals to the n line segments.

Now suppose that all the $n \geqslant 3$ supporting lines of the segments s_i lie in the first ruling of a hyperbolic paraboloid. The lines in the second ruling can be parameterized by their intersection points with any line r of the first ruling. Thus the set of lines in the second ruling that meet a segment s_i corresponds to an interval on line r. Hence the set of transversals to the n segments corresponds to the intersection of n intervals

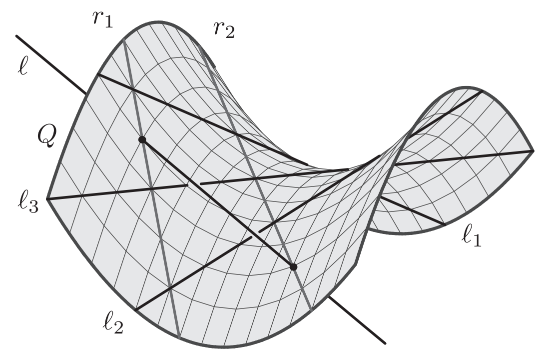

Figure 2: Line ℓ intersects in two points the hyperbolic paraboloid spanned by the lines ℓ_1, ℓ_2 and ℓ_3. The two lines r_1 and r_2 meet the four lines ℓ_1, ℓ_2, ℓ_3, and ℓ.

on r, that is, to one interval on this line, and so the transversals form one connected component.

Consider finally the case where the $n \geqslant 3$ supporting lines lie in the first ruling of a hyperboloid of one sheet (see Figure 1). The lines in the second ruling can be parameterized by points on a circle, for instance, by their intersection points with a circle lying on the hyperboloid of one sheet. Thus the set of transversals to the n segments corresponds to the intersection of n intervals on this circle. This intersection can have any number of connected components between 0 and n and any of these connected components may consist of an isolated point on the circle. The set of transversals can thus have any number of connected components between 0 and n and any of these connected components may consist of an isolated transversal. Figure 1 shows two views of a configuration with $n = 4$ line segments having 4 connected components of transversals.

In this section we have proved that if the supporting lines of $n \geqslant 3$ line segments lie in one ruling of a hyperboloid of one sheet, the segments admit $0, 1, \ldots, n$ or infinitely many transversals which form up to n connected components. If supporting lines lie in one ruling of a hyperbolic paraboloid, the segments admit at most 1 connected component of transversals. Otherwise the segments admit up to 2 transversals.

2.2 Two supporting lines are coplanar

Let ℓ_1 and ℓ_2 be two (distinct) coplanar supporting lines in a plane H. First consider the case where ℓ_1 and ℓ_2 are parallel. Then the transversals to the n segments all lie in H. If some segment does not intersect H then there are no transversals; otherwise, we can replace each segment by its intersection with H to obtain a set of coplanar segments, a configuration treated in Section 2.3.

Now suppose that ℓ_1 and ℓ_2 intersect at point p. Consider all the supporting lines not in H. If no such line exists then all segments are coplanar; see Section 2.3. If such lines exist and any of them is parallel to H then all transversals to the n segments lie in the plane containing p and that line. We can again replace each segment by its intersection with that plane to obtain a set of coplanar segments, a configuration treated in Section 2.3.

We can now assume that there exists a supporting line not in H. Suppose that all the supporting lines not in H go through p. If all the segments lying in these supporting lines contain p then we may replace all these segments by the point p without changing the set of transversals to the n segments. Then all resulting segments are coplanar, a configuration treated in Section 2.3. Now if some segment s does not contain p then the only possible transversal to the n segments is the line containing s and p.

We can now assume that there exists a supporting line ℓ_3 intersecting H in exactly one point q distinct from p (see Figure 3). Let K be the plane containing p and ℓ_3. Any transversal to the lines ℓ_1, ℓ_2 and ℓ_3 lies in K and goes through p, or lies in H and goes through q.

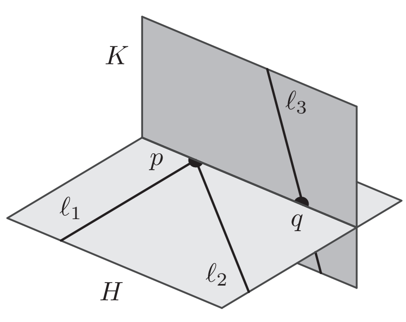

Figure 3: Lines ℓ_1 and ℓ_2 intersect at point p, and line ℓ_3 intersects plane H in a point q distinct from p.

If there exists a segment s that lies neither in H nor in K and goes through neither p nor q, then there are at most two transversals to the n segments, namely, at most one line in K through p and s and at most one line in H through q and s.

We can thus assume that all segments lie in H or K or go through p or q. If there exists a segment s that goes through neither p nor q, it lies in H or K. If it lies in H then all the transversals to the n segments lie in H (see Figure 4). Indeed, no line in K through p intersects s except possibly the line pq which also lies in H. We can again replace each segment by its intersection with H to obtain a set of coplanar segments; see Section 2.3. The case where s lies in K is similar.

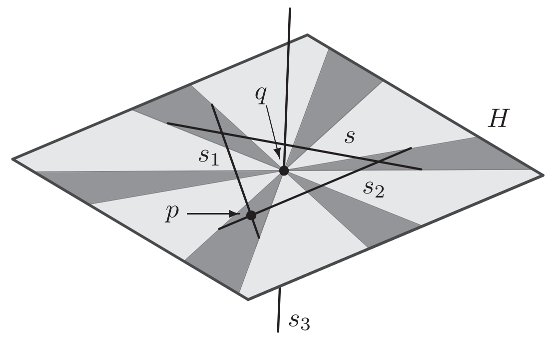

Figure 4: Four segments having three connected components of transversals.

We can now assume that all segments go through p or q (or both). Let n_p be the number of segments not containing p, and n_q be the number of segments not containing q. Note that $n_p + n_q \leqslant n$.

Amongst the lines in H through q, the transversals to the n segments are the transversals to the n_q segments not containing q. We can replace these n_q segments by their intersections with H to obtain a set of n_q coplanar segments in H. The transversals to these segments in H through q can form up to n_q connected components. Indeed, the lines in H through q can be parameterized by a point on a circle, for instance, by their polar angle in $\mathbb{R}/\pi\mathbb{Z}$. Thus the set of lines in H through q and through a segment in H corresponds to an interval of $\mathbb{R}/\pi\mathbb{Z}$. Hence the set of transversals to the n_q segments corresponds to the intersection of n_q intervals in $\mathbb{R}/\pi\mathbb{Z}$ which can have up to n_q connected components.

Similarly, the lines in K through p that are transversals to the n segments can form up to n_p connected components. Note furthermore that the line pq is a transversal to all segments and that the connected component of transversals that contains the line pq is counted twice. Hence there are at most $n_p + n_q - 1 \leqslant n - 1$ connected components of transversals to the n segments.

To see that the bound of $n - 1$ connected components is reached, first consider $n/2$ lines in H through p, but not through q. Their transversals through q are all the lines in H though q, except for the lines that are parallel to any of the $n/2$ given lines. This gives $n/2$ connected components. Shrinking the $n/2$ lines to

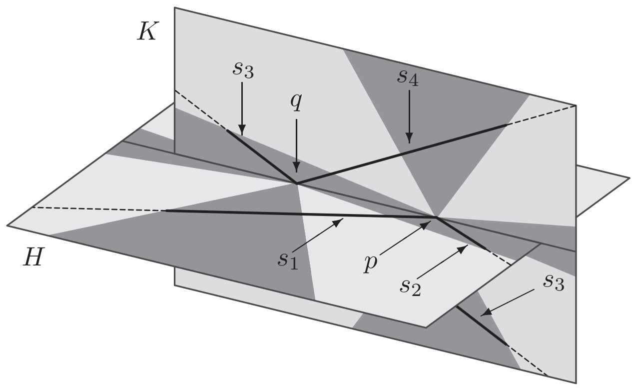

Figure 5: Four segments having three connected components of transversals.

sufficiently long segments still gives $n/2$ connected components of transversals in H through q. The same construction in plane K gives $n/2$ connected components of transversals in K through p. This gives $n-1$ connected components of transversals to the n segments since the component containing line pq is counted twice. Figure 5 shows an example of 4 segments having 3 connected components of transversals.

In this section we have proved that $n \geqslant 3$ segments having at least two coplanar supporting lines either can be reduced to n coplanar segments or may have up to $n-1$ connected components of transversals.

2.3 All the line segments are coplanar

We prove here that $n \geqslant 3$ coplanar line segments in $\mathbb{R}^3$ admit up to n connected components of transversals.

Let H be the plane containing all the n segments. There exists a transversal not in H if and only if all segments are concurrent at a point p. In this case, the transversals consist of the lines through p together with the transversals lying in H. To see that they form only one connected component, notice that any transversal in H can be translated to p while remaining a transversal throughout the translation. We thus can assume in the following that all transversals lie in H, and we consider the problem in $\mathbb{R}^2$.

We consider the usual geometric transform (see e.g. [1]) where a line in $\mathbb{R}^2$ with equation $y = ax + b$ is mapped to the point (a, b) in the dual space. The transversals to a segment are transformed to a double wedge; the double wedge degenerates to a line when the segment is a point. The apex of the double wedge is the dual of the line containing the segment.

A transversal to the n segments is represented in the dual by a point in the intersection of all the double wedges. There are at most $n+1$ connected components of such points [1]. Indeed, each double wedge consists of two wedges separated by the vertical line through the apex. The intersection of all the double wedges thus consists of at most $n+1$ convex regions whose interiors are separated by at most n vertical lines.

Notice that if there are exactly $n+1$ convex regions then two of these regions are connected at infinity by the dual of some vertical line, in which case the segments have a vertical transversal. Thus the number of connected components of transversals is at most n.

To see that this bound is sharp consider the configuration in Figure 6 of 4 segments having 4 components of transversals. Three of the components consist of isolated lines and one consists of a connected set of lines through p (shaded in the figure). Observe that the line segment ab meets the three isolated lines. Thus the set of transversals to the four initial segments and segment ab consists of the 3 previously mentioned isolated transversals, the line pb which is isolated, and a connected set of lines through p. This may be repeated for any number of additional segments, giving configurations of n coplanar line segments with n connected components of transversals.

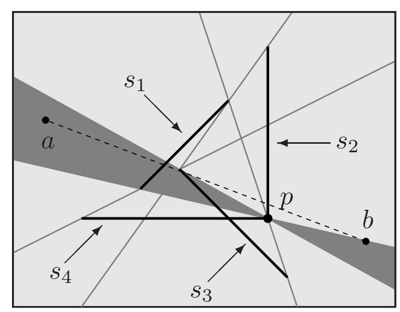

Figure 6: Four coplanar segments having four connected components of transversals.

Acknowledgments

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