Hinged Dissection of Polygons is Hard

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Abstract

We show several natural questions about hinged dissections of polygons to be PSPACE-hard. The most basic of these is: Given a hinged set of pieces and two configurations for them, can we swing the pieces on the hinges to transform one configuration to the other? We also consider variants in which the configurations must be convex, the placement of hinges is not specified, or the configurations are not supplied, but just the target shapes. We show all of these variants to be PSPACE-hard, via a reduction from Nondeterministic Constraint Logic [4].

1 Introduction

Geometric dissection problems have a long, colorful history, reaching back to the ancient Greeks and to the golden age of Islamic civilization. Approximately a century ago, Henry Dudeney and Sam Loyd championed them in mathematical puzzle columns, and they have enjoyed increasing popularity ever since [2].

It was Dudeney who pointed out that the 4-piece dissection of an equilateral triangle to a square is hingeable. (See Fig. 1.) Over the years, other hinged dissections have been noted [2], and recently a whole book on the subject has appeared [3]. In contrast to the situation for unhinged dissections, it is not known whether for any given polygons of equal area, there is a hinged dissection. Even when given a dissection and an associated assignment of hinges, testing whether there is a continuous motion transforming the one polygon to the other seems to be difficult. This paper confirms this notion of hardness, showing that this associated motion planning problem and the other variants are PSPACE-hard.

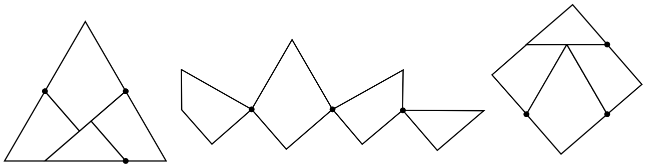

Figure 1: Hinged triangle-to-square dissection.

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Terminology. We define a *piece* as an instance of a polygon. A *dissection* is a set of pieces. A *configuration* is an embedding of a dissection in the plane, such that only the boundaries of pieces may overlap. The *shape* of a configuration is the set of points, including the boundaries of pieces, that it occupies.

A *hinge point* of a piece is a given point on the boundary of the piece. A *hinge* for a set of pieces is a set of hinge points, one for each piece in the set. Given a dissection and a set of hinges, two pieces are *hinge-connected* if either they share a hinge or there is another piece in the set to which both are hinge-connected. A *hinging* of a dissection is a set of hinges such that all pieces in the dissection are hinge-connected.

A *hinged configuration* of a dissection and a hinging is a configuration that, for each hinge, collocates all hinge points of that hinge. A *kinematic solution* of a dissection, a hinging, and two hinged configurations is a continuous path from one configuration to the other through the space of hinged configurations.

Nondeterministic Constraint Logic. We show that our hinged-dissection problems are PSPACE-hard by a reduction from Nondeterministic Constraint Logic (NCL) [4].

An NCL "machine" is specified by a *constraint graph*: an undirected planar graph together with an assignment of weights from $\{1, 2\}$ to each edge. A configuration of this machine is an orientation (direction) of the edges such that the sum of incoming edge weights at each vertex is at least 2. A move is made by reversing a single edge such that the configuration remains valid.

The following decision questions are PSPACE-complete: (1) starting from a specified configuration, can another specified configuration be reached by a sequence of moves ("configuration to configuration"), and (2) given edges E_A and E_B with desired orientations, do there exist configurations A, with E_A in its desired orientation, and B, with E_B in its desired orientation, such that B can be reached from A ("edge to edge").

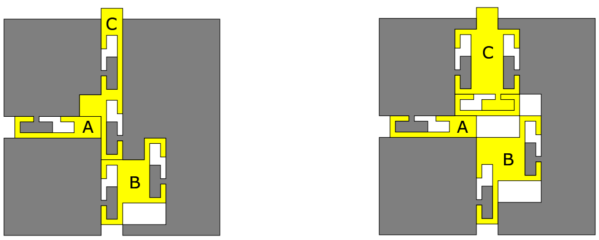
In fact, only two types of vertices are necessary for PSPACE-completeness to hold: those with incident edge weights of 1-1-2 ("AND") and 2-2-2 ("OR"). These vertex types have properties similar to the logical operations of the same name. For example, for the weight-2 edge to be directed away from an AND vertex, both of the weight-1 edges must be directed inward.

2 PSPACE-hardness

We begin with the most basic decision question: given a dissection, a hinging, and two hinged configurations, does there exist a kinematic solution? After showing this problem to be PSPACE-hard, we generalize the result to apply to the other questions as well.

2.1 Basic Construction

We show how to form NCL AND and OR vertices from partial configurations, which can be assembled to form hinged configurations corresponding to given NCL graph configurations.



(a) AND vertex (b) OR vertex

Figure 2: NCL vertex gadgets.

Vertex gadgets. Fig. 2 shows the vertex gadgets. Each gadget is made of several pieces. The pieces are joined together by hinges; each sliding connection in Fig. 2 is shorthand for a hinged connection as shown in Fig. 3.

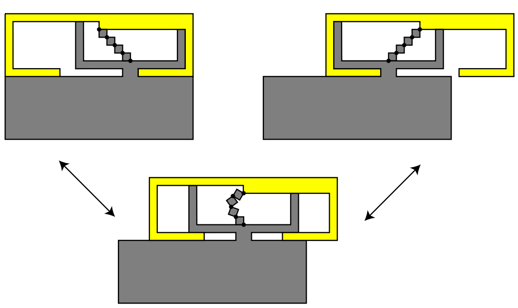


Figure 3: Hinged slider.

In each vertex gadget, a sliding piece slid into the gadget (A and B) represents an edge directed *outward* from the vertex; a piece protruding from the gadget (C) represents an edge directed *into* the vertex.

Planar graph configurations. We combine the vertex gadgets into a configuration corresponding to a given NCL graph. We simply tile a square grid with the gadgets as needed, joining adjacent framework pieces together. The resulting construction goes inside a single hollow square piece, to ground the framework and keep the framework pieces from moving.

Lemma 1 *The partial configuration in Fig. 2(a) satisfies the same constraints as an NCL AND vertex.*

Proof sketch: Piece C can slide into the gadget if and only if pieces A and B first slide out. This corresponds to the edge redirection constraints in an AND vertex: the weight-2 edge can be directed out if and only if both weight-1 edges are directed in. □

Lemma 2 *The partial configuration in Fig. 2(b) satisfies the same constraints as an NCL OR vertex.*

Theorem 3 *Given a dissection, a hinging, and two hinged configurations, determining whether there is a kinematic solution is PSPACE-hard.*

Proof: Given source and target NCL configurations, we form corresponding hinged configurations, as described above. Each vertex gadget edge piece corresponds to half an edge in the graph; for a half edge to be slid outward from a vertex gadget, its matching half edge must be slid into its own gadget. This property ensures that a kinematic solution exists if and only if the target NCL configuration is reachable from the source. □

2.2 No Hinging is Given

Theorem 4 *The following question is PSPACE-hard: Given a dissection and two of its configurations, do a hinging and a kinematic solution exist?*

Proof: We use the construction in Sec. 2.1, without the hinges. The construction gains no essential freedom of motion if no hinges are specified. Therefore, if a solution to the NCL problem exists, the original hinging suffices; if no NCL solution exists, no hinging will yield a kinematic solution. □

2.3 Convex Configuration Shapes

If the source and target configuration shapes are required to be convex, does the problem get easier? No, as we show in stages.

We begin by eliminating the internal holes from the configuration described in Sec. 2.1. We introduce gaps between all the pieces that are not directly joined by hinges, and also introduce a tree of gaps along vertex boundaries, to reach all the vertex entrances. We connect the resulting single empty space to the exterior of the enclosing square via a narrow tunnel.We can easily make the gaps small enough not to affect the vertex behavior. We introduce the gaps in such a way as to cause the empty space to form a polyomino-shaped hole. If we can find a single hinged dissection which can kinematically fill any m-omino-shaped hole, for fixed m, we can make the configuration shapes convex.

A hinged dissection which can form any m-omino appears in [1]; this construction will not suffice, however, for kinematically filling and emptying holes. We introduce a dissection which solves this problem.

First, we give a dissection (isosceles dissection #1) permitting a right isosceles triangle to be flipped, while holding the vertices at its acute angles (Fig. 4). Using isosceles dissection #1 four times, we next form a dissection (isosceles dissection #2) permitting a right isosceles triangle to be flipped while holding the other possible vertex pair (Fig. 5). Using isosceles dissection #2 twice, we form a dissection (universal isosceles dissection) among all six possible orientations of a right isosceles triangle, holding any two vertices (Fig. 6).

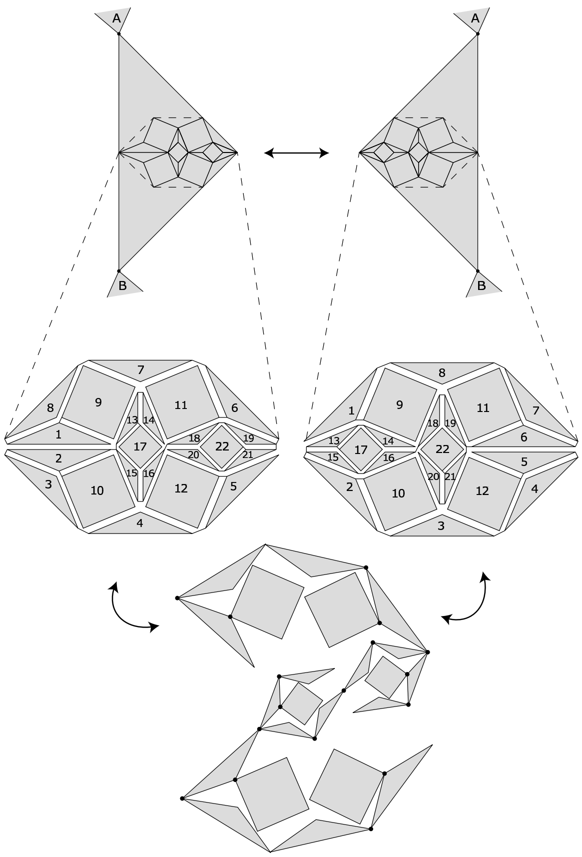


Figure 4: Isosceles dissection #1.

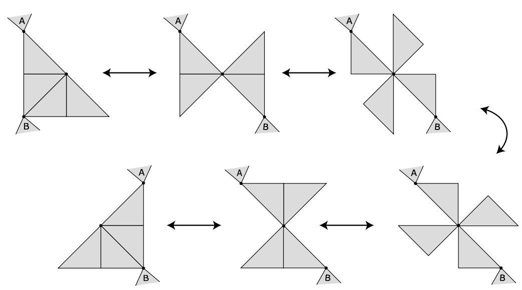


Figure 5: Isosceles dissection #2.

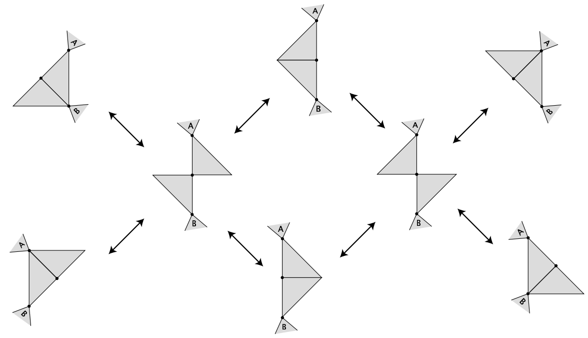


Figure 6: Universal isosceles dissection.

We use all these dissections to fill polyomino-shaped holes. Using strings of eight universal isosceles dissections, we form the patterns *descent*, *turn 1*, and *turn 2* (Fig. 7). We can form a square in two different ways (Fig. 8). Each square may be unfolded by pulling the triangle strings out from above, without them occupying any space otherwise adjacent to the square. After the string has been extracted, in Fig. 8(a) (*straight*) the tail of the string touches the bottom center of the square; in Fig. 8(b) (*turn*) the tail touches the right center.

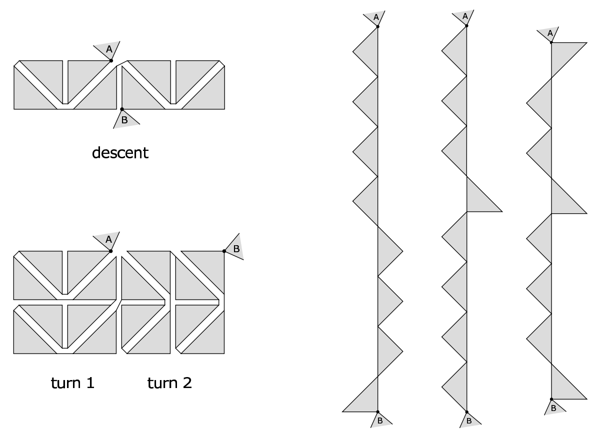
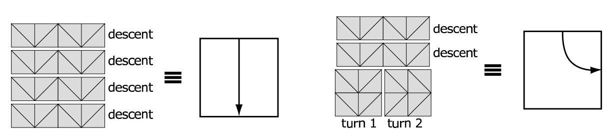


Figure 7: Descent, turn 1, turn 2: folded, straight



(a) Straight (b) Turn

Figure 8: Hole-filling primitives.

By choosing a suitable string of the primitive patterns, we can form any sequence of straight and turn squares. We then use this string to fill a polyomino-shaped hole (Fig. 9). Imagine that the tail of the string is at B, attached to the surrounding piece framework. If the hole is filled in by the dissection, we may extract it completely from the surrounding shape by pulling out one square at a time, as described above, beginning at

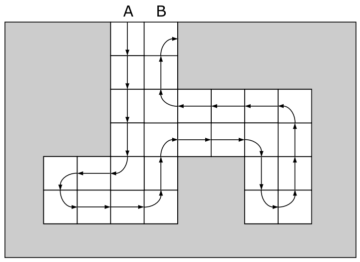


Figure 9: How to fill a polyomino-shaped hole.

A. Reversing this procedure fills the hole.

Since we have used universal isosceles dissections to form the triangles in Fig. 7, we can reconfigure the hole-filling string, while extracted from the shape, to form any other hole-filling string: we can switch between descent, turn 1, and turn 2, or reflected versions of the turns, for each segment. This lets a single string fill *any* m-omino-shaped hole, for fixed m.

Theorem 5 *Theorem 3 holds even when the configurations are required to form convex shapes.*

Proof: Proof omitted in this abstract. □

2.4 The Configurations are not Given

For Theorems 3 and 5 we are given two configurations of a dissection, and the problem is to find a kinematic solution between them. What if the configurations are unspecified, and instead we are given merely the shapes that the configurations must form? For example, for Dudeney's triangle-to-square dissection (Fig. 1), we might be given the set of pieces and hinges, and the triangle and square shapes, and asked whether there are any hinged configurations yielding the shapes, such that a kinematic solution exists between them. In this section we show that these problems are also PSPACE-hard, even when the shapes are rectangles.

Forced configuration properties. First note that the dissection and hinging given in Sec. 2.1 force all hinged configurations to correspond to the same NCL machine, but with different NCL configurations, depending on the slider positions. The chain of squares in Fig. 3 forces each pair of slider pieces to mate properly; the left-right asymmetry in the slider ensures that a flipped configuration is not possible.

Rectangle-to-rectangle construction. We extend the original construction with extra pieces, as shown in Fig. 10. The original square construction is embedded in this construction as block M. We use the edge-to-edge NCL decision problem, and arrange for piece A to be able to slide into M only when one input edge is in its desired orientation, and for piece B to be able to

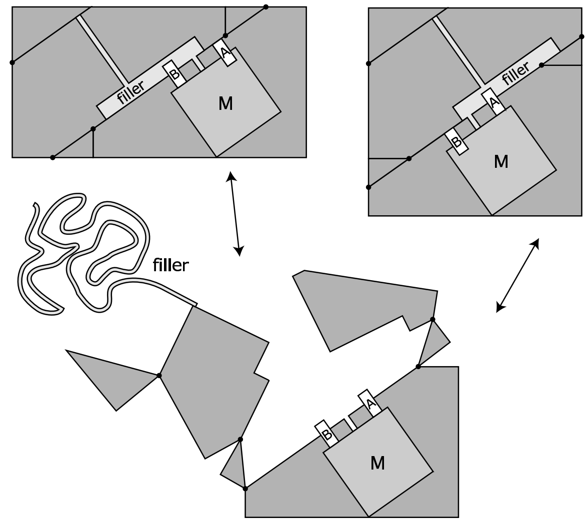


Figure 10: Rectangle-to-rectangle construction.

slide into M only when the other input edge is in its desired orientation. (A and B are both connected to the framework pieces with hinged sliders.)

As in Sec. 2.3, we fill the empty space in M with a universal m-omino dissection. We increase m by a sufficient amount to include the space marked "filler" in the figure.

Based on the positions of A and B, the configuration can form one of the two rectangular shapes, as required.

Theorem 6 *Given a dissection, a hinging, and two rectangles, determining whether there are hinged configurations admitting a kinematic solution is PSPACE-hard.*

Proof: Proof omitted in this abstract. □

3 Conclusion

Our results can be characterized according to four dimensions:

1. whether the hinging is given, or it is free to be chosen;
2. whether the source and target are specified as configurations, or as shapes;
3. whether the source and target shapes are rectangles, convex polygons, or general polygons; and
4. whether the pieces are restricted to be convex.

When the hinging is given (1), our theorems apply no matter how the source and target are specified (2) and even when the shapes are rectangles (3) and the pieces are convex (4) (convex piece discussion omitted from this abstract). However, when the hinging is free (1), our theorems apply only when the source and target are specified as configurations (2) and the shapes are general polygons (3), although the pieces may be convex (4). The remaining problems in this 4D space are open, and it seems difficult to adapt our reduction to handle them.

References

[1] Erik D. Demaine, Martin L. Demaine, David Eppstein, Greg N. Frederickson, and Erich Friedman. Hinged dissection of polyominoes and polyforms. In *Computational Geometry: Theory and Applications*, to appear.

[2] Greg N. Frederickson. *Dissections Plane & Fancy*. Cambridge University Press, New York, 1997.

[3] Greg N. Frederickson. *Hinged Dissections: Swinging and Twisting*. Cambridge University Press, New York, 2002.

[4] Robert A. Hearn and Erik D. Demaine. The Nondeterministic Constraint Logic model of computation: Reductions and applications. In Proc. 29th Int. Colloq. Automata, Languages, and Programming (2002).