Computing Delaunay Triangulation with Imprecise Input Data

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Abstract

The key step in the construction of the Delaunay triangulation of a finite set of planar points is to establish correctly whether a given point of this set is inside or outside the circle determined by any other three points. We address the problem of formulating the in-circle test when the coordinates of the planar points are given only up to a given precision, which is usually the case in practice. By modelling imprecise points as rectangles, and using the idea of partial disc, we construct a reliable in-circle test that provides the best possible Delaunay triangulation with the imprecise input data given by rectangles.

keywords: *robust algorithm, Delaunay triangulation, imprecise data, in-circle test, partial disc.*

1 Introduction

A Delaunay triangulation of a set of N planar points is a triangulation with the property that the interior of the circumcircle of each triangle contains no points of the set [3]. If there are no four cocircular points in the set, then the Delaunay Triangulation is unique. A divide and conquer algorithm [8] and an algorithm based on sweepline technique [7] compute the Delaunay Triangulation in $O(N \log N)$, while a random incremental algorithm [3] has an average complexity of $O(N \log N)$. We present the simplest algorithm [12], which is $O(N^4)$, but illustrates the crucial role of the so-called in-circle test in computing the Delaunay Triangulation:

Algorithm:

```
for each of the possible N choose 3 "triangles" of points do
{
   for each of the other N - 3 points do
   {
      if all the other points are outside of the circle
         circumscribing the triangle (i.e. THE IN-CIRCLE TEST)
      then this triangle is in the unique Delaunay triangulation
      else this triangle is not in the unique Delaunay triangulation
   }
}
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Any algorithm for Delaunay Triangulation is based on correctly establishing whether a given point is inside or outside the circumcircle of three given points. When the coordinates of the N points are infinitely precise, this in-circle test is determined in terms of finding the signs of various determinants. The Delaunay triangulation is degenerate, i.e. is not unique, if at least one of these determinants is itself zero or is computed to zero, given the precision of computation. In [13], degenerate cases in Delaunay triangulation are removed by modifying the input data, whereas in [1] a tolerance value for the input points is obtained, within which degenerate cases do not occur.

In practice, the input data is usually imprecise as the coordinates of the points are either the result of some measurements with certain error or are actually floating point numbers which represent rectangles rather than points in the plane. The in-circle test then becomes non-trivial.

Ely and Leclerc [6] have formulated the following two essential properties for the in-circle test. By *reliable* it is meant that the test must never make an error if it does decide that the point is inside or outside the circle. By *sharp* it is meant that the test should only fail (to decide whether the point is inside or outside of the circle) because of the underlying geometry of the imprecise points.

The use of interval arithmetic [11] can provide *reliability* by ensuring that the successful result of the in-circle test is always correct, but this is almost always at the expense of sharpness as the test may frequently fail unjustifiably.

In [6], Ely and Leclerc model an imprecise point by a disc and show that any three discs representing three imprecise points determine eight circles in the plane, each tangent to the three discs, which are then used to formulate an in-circle test that is sharper than the naive interval arithmetic.

However, in practice, an imprecise point is almost always given by two imprecise x and y coordinates, representing two horizontal and vertical line segments respectively, which therefore give rise to a rectangle rather than a disc.

In [10], the notion of partial Delaunay triangulation and the partial Voronoi diagram of N partial points, represented by N rectangles, in the plane were introduced and shown to be effectively computable in an order-theoretic framework for computational geometry [4]. This approach, i.e. representing the imprecise points by rectangles, has already been used in solving the problem of computing the convex hull for imprecise input [5], which computes an interior and an exterior for the partial convex hull. As the rectangles converge the exact points, the interior and the exterior of the partial convex hull converge to the interior and the exterior of the classical convex hull for the exact points.

In this paper, following the above approach, we represent imprecise points by rectangles and, using the notion of partial disc established in [10], we construct a reliable in-circle test which is completely sharp: if the output is neither "inside" nor "outside", then we have possible degeneracy with the given impreciseness of the input, i.e. there exist four cocircular points, one in each of four rectangles.

In the classical case, the Delaunay triangulation of N points is equivalent to the Voronoi diagram of these N points (sites) [3, 2]. Given the Delaunay triangulation of the N points, one can find the Voronoi diagram by drawing the perpendicular bisectors of all the edges of the triangles. Conversely, given the Voronoi diagram of the N points, the Delaunay triangulation can be constructed by connecting the sites in each pair of neighbouring regions. Therefore, solving either of these problems will also solve the other. But we note that the problem of Delaunay triangulation with imprecise input, addressed here, is equivalent to the partial Voronoi diagram considered in [10], which is different from the problem of the Voronoi diagram of N rectangles in the plane. In the problem of Voronoi diagram of N rectangles, one has to determine N regions of the plane, each containing exactly one of the rectangles, such that the rectangle in that region is the closest rectangle to any point in that region. In fact, these two problems, i.e. partial Voronoi diagram and Voronoi diagram for rectangles, are based on different distance functions as explained in [10]. See also [9] for the similar case of the Voronoi diagram of finite number of line segments in the plane.

2 Partial Disc

We first recall the notion of a partial disc [10], which is the generalization of a disc. Any three noncollinear points x, y, z in the plane determine a disc D_{xyz} which has x, y, z on its boundary. Now assume that all we know is that the three points x, y, z lie respectively in the three rectangles R_1, R_2, R_3, where each rectangle $R = [a, b] \times [c, d]$ is the product of two rational intervals $[a, b]$ and $[c, d]$. We say that the three rectangles are *noncollinear* if there is no straight line which intersects all three rectangles. Assuming the noncollinearity condition, R_1, R_2, R_3 will determine a *partial disc* that is given by a pair of disjoint open subsets $(I(R_1, R_2, R_3), E(R_1, R_2, R_3))$ with the *interior*

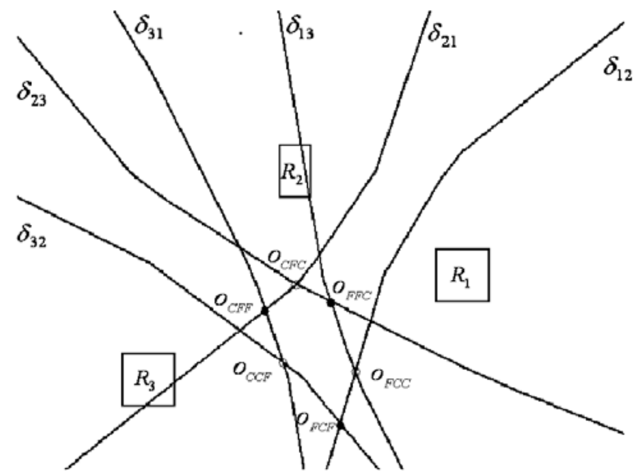

Figure 1: The six centres

$$I(R_1, R_2, R_3) = (\bigcap\{D_{xyz} \mid x \in R_1,\ y \in R_2,\ z \in R_3\})^\circ,$$

and the *exterior*

$$E(R_1, R_2, R_3) = (\bigcup\{D_{xyz} \mid x \in R_1,\ y \in R_2,\ z \in R_3\})^c,$$

where A° and A^c are respectively the interior and complement of the set A. In words, $I := I(R_1, R_2, R_3)$ is the largest open set contained in the interior of any circle which intersects all three rectangles R_1, R_2, R_3, whereas $E := E(R_1, R_2, R_3)$ is the largest open set contained in the exterior of any such circle.

The sets I and E are computed as follows. Let δ_{ij} be the locus of all points in $\mathbb{R}^2$ with the furthest distance from points in R_i equal to the closest distance to R_j, i.e.

$$\delta_{ij} = \{x \in \mathbb{R}^2 \mid \max_{p \in R_i} d(x, p) = \min_{p \in R_j} d(x, p)\}.$$

Each δ_{ij} is a continuous curve made up of a number of linear and parabolic segments as in Figure 1.

Proposition 2.1 *The following intersections*

$$\delta_{12} \cap \delta_{13},\ \delta_{21} \cap \delta_{23},\ \delta_{31} \cap \delta_{32},\ \delta_{21} \cap \delta_{31},\ \delta_{12} \cap \delta_{32},\ \delta_{13} \cap \delta_{23}$$

are either singleton or empty. They are all singletons iff the three rectangles are noncollinear, otherwise some of these intersections will be empty.

Assuming the noncollinearity condition, we define six centre points as follows:

$$\begin{aligned} \{o_{FCC}\} &= \delta_{12} \cap \delta_{13}, & \{o_{CFF}\} &= \delta_{21} \cap \delta_{31}, \\ \{o_{CFC}\} &= \delta_{21} \cap \delta_{23}, & \{o_{FCF}\} &= \delta_{12} \cap \delta_{32}, \\ \{o_{CCF}\} &= \delta_{31} \cap \delta_{32}, & \{o_{FFC}\} &= \delta_{13} \cap \delta_{23}. \end{aligned} \tag{1}$$

These points have been depicted in Figure 1. Note that o_{CCF} is the centre of a circle with radius r_{CCF}, which passes through these three points:

(i) the point of R_1 closest to o_{CCF},

(ii) the point of R_2 closest to o_{CCF},

(iii) the point of R_3 furthest from o_{CCF},

hence the subscript in o_{CCF}, and similarly for the other vertices. If any one or more of the rectangles R_1, R_2 and R_3 are in fact singletons then some of the above centre points will coincide.

Let $D(o, r)$ denote the closed disc with centre o and radius r. The interior I and the exterior E of the partial disc of R_1, R_2, R_3 are given by:

$$I = (D_1 \cap D_2 \cap D_3)^\circ, \qquad E = (D'_1 \cup D'_2 \cup D'_3)^c,$$

where

$$\begin{aligned} D_1 &= D(o_{FCC}, r_{FCC}), & D'_1 &= D(o_{CFF}, r_{CFF}), \\ D_2 &= D(o_{CFC}, r_{CFC}), & D'_2 &= D(o_{FCF}, r_{FCF}), \\ D_3 &= D(o_{CCF}, r_{CCF}), & D'_3 &= D(o_{FFC}, r_{FFC}). \end{aligned}$$

In Figure 2, the boundaries of the discs D_1, D_2 and D_3 are depicted with dotted lines, those of D'_1, D'_2 and D'_3 with solid lines and the boundaries of the sets I and E with dashed lines. The closed region bounded between I and E, i.e. bounded between the two closed dashed curves, is the *boundary of the partial disc*.

Note that there are two other discs, one which passes through the closest point of each of the three rectangles and one which passes through the furthest points of these rectangles. The interior of the partial disc is contained in the interiors of these two discs, and also the exterior of the partial disc is contained in the exteriors of these two discs. Thus, they lie completely inside the boundary of the partial disc and do not contribute any extra information for the construction of the partial disc.

3 The In-Circle Test

We start by first computing the δ_{ij} boundaries as in [10]. Then, we find the intersection points as in (1), which give us the centres of the circles. When all the three imprecise points are non-degenerate rectangles, there will be three centres for the interior circles and three centres for the exterior circles. The following formula gives the radius r of any of the six circles, which make up the

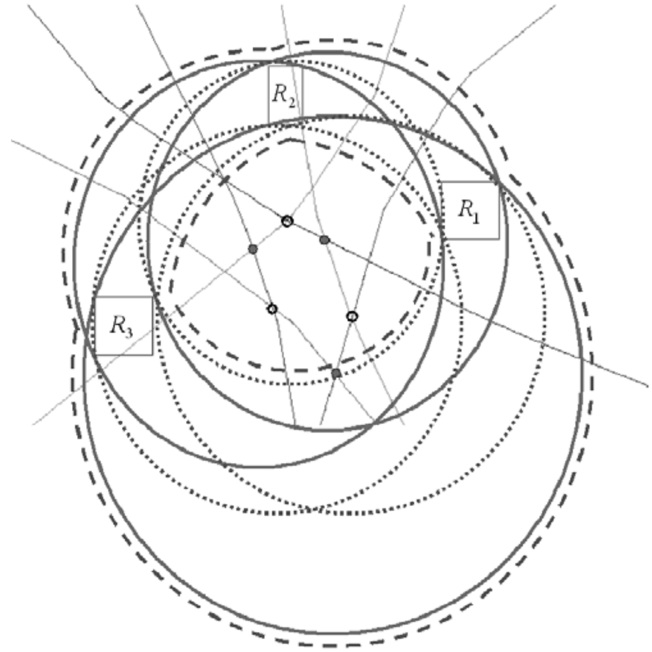

Figure 2: The interior and the exterior of a partial disc

interior and the exterior of the partial disc, in terms of its centre o:

$$r = \min \{ \{ \max d(o, p) \mid p \text{ is a vertex of } R_i \} \mid i = 1, 2, 3 \} .$$

Using the following *containment* predicate, we can verify that a given imprecise point or rectangle R is inside, outside, or on the boundary of a partial disc given by three rectangles R_1, R_2, R_3.

$$\mathrm{Con}(R, (R_1, R_2, R_3)) = \begin{cases} \text{inside} & \text{if } R \subset I(R_1, R_2, R_3) \\ \text{outside} & \text{if } R \subset E(R_1, R_2, R_3) \\ \text{boundary} & \text{otherwise.} \end{cases}$$

Since the coordinates of the centres of the six interior and exterior circles are intersections of straight lines or parabolas with rational coefficients, the output of the containment predicate can be correctly computed using rational arithmetic.

Theorem 3.1 *If the three rectangles are collinear, then the in-circle algorithm will verify it using Proposition 2.1. Otherwise, with the noncollinearity condition, the algorithm outputs the correct value of the containment predicate.*

We finally note that computing the partial disc passing through three imprecise points and applying the in-circle test for the fourth imprecise point is of order $O(1)$ and does not affect the complexity of any algorithm which uses this test.

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