Equiprojective Polyhedra

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Abstract

A convex polyhedron P is *equiprojective* if, for some k, the orthogonal projection (or "shadow") of P in every direction, except those directions parallel to faces of P, is a k-gon. We address an open question posed by Shepherd [11], and reported in [5], by characterizing equiprojective polyhedra, and giving an $O(n \log n)$ time recognition algorithm.

1 Introduction

A convex polyhedron P is k-*equiprojective* if its shadow is a k-gon in every direction, except directions parallel to faces of P. A cube is 6-equiprojective, a triangular prism is 5-equiprojective, and a tetrahedron is not equiprojective. See Figure 1.

In 1968 Shepherd [11] defined equiprojective polyhedra, gave the examples above, and asked how to construct all equiprojective polyhedra. Croft, Falconer, and Guy include this problem in their book [5].

We note that the cube and triangular prism can be generalized: for any $p \geq 3$, a prism based on a regular p-gon is $(p+2)$-equiprojective. An example of equiprojective polyhedron that is not a prism is given in Figure 2.

In this paper we give a characterization of equiprojective polyhedra, and show that this characterization provides an $O(n \log n)$ time algorithm to test if a polyhedron of size n is equiprojective.

Our characterization can be used to show that all generalized zonohedra are equiprojective, and we identify other interesting subclasses as well. See Section 2. The whole class seems surprisingly rich, and we do not give a method for generating it.

The flavour of our characterization is as follows. Any edge, e, of the shadow of P corresponds to some edge of P. As the projection direction changes, e may leave the shadow boundary. This only happens when a face f containing e in P becomes parallel to the direction. In order to preserve the size of the shadow, some other edge e' must join the shadow boundary. In order for these events to occur simultaneously, e' must be an edge of f, or of a face parallel to f. This gives some intuition that the condition for equiprojectivity involves a pairing-up of parallel edge-face pairs of P. For a more precise statement of our characterization, see Section 2.

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Background

One way to test if a polyhedron is equiprojective would be to check all the combinatorially different projections, and for each one count the number of edges of the shadow. Plantinga and Dyer [10] show that the number of combinatorially different orthographic projections of an n vertex convex polyhedron is $O(n^2)$. (In their terminology, this is the size of the viewpoint space partition.) This method of testing for equiprojectivity is thus polynomial time, though inefficient. It might be improved if we could quickly find the projections (not parallel to faces) with the minimum and maximum number of edges. We know of no such algorithmic results, though there are bounds known on the asymptotic size of the 2-dimensional projection of a d-dimensional polytope [1].

When the size of the shadow is measured in terms of area rather than number of vertices, there are algorithmic results on the problem of maximizing or minimizing the size of the shadow. For projections to one dimension, this is the problem of finding the diameter and the width of a convex polyhedron. In the case of higher-dimensional polyhedra we measure the volume of the projection. See [3] for a mathematical treatment; [9] for an algorithm for the case of 3-dimensional polytopes; and [2] for an NP-completeness proof for higher-dimensional versions of the problem.

Projections of objects consisting only of edges, not planar faces, have also been explored. For example, there are methods for finding projections of knots and embedded graphs that avoid unnecessary incidences. See [13] for a survey.

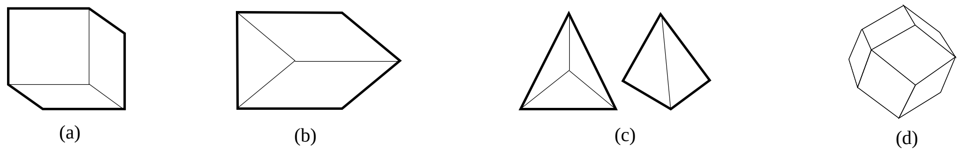

Figure 1: (a) A cube is 6-equiprojective, (b) a triangular prism is 5-equiprojective, (c) a tetrahedron is not equiprojective, and (d) a zonohedron, which we discuss in Section 2, is always equiprojective.

2 Which polyhedra are equiprojective?

We begin with the precise statement of our characterization, and then explore some classes of equiprojective polyhedra.

For edge e in face f, we call (e, f) an *edge-face duple*. Two edge-face duples (e, f) and (e', f') are *parallel* if e is parallel to e' and f is parallel to, or equal to f'. Observe that in a convex polygon, an edge can have at most one parallel edge; and in a convex polyhedron, a face can have at most one parallel face. Thus an edge-face duple has at most three parallel duples.

Define the *direction* of duple (e, f) to be a unit vector in the direction of edge e as encountered in a clockwise traversal of the outside of face f. Edge-face duples (e, f) and (e', f') *compensate* each other if they are parallel and their directions are opposite (i.e. one is the negation of the other). In particular, this means that either $f = f'$ and e and e' are parallel (in which case they must be on "opposite sides" of f), or f and f' are distinct parallel faces, and e and e' are parallel edges lying on the "same side" of f and f'. See Figure 2. An edge-face duple has at most two compensating duples.

Theorem 2.1 *Polyhedron P is equiprojective iff its set of edge-face duples can be partitioned into compensating pairs.*

This theorem is proved in Section 3.

One of the simplest subclasses of equiprojective polyhedra are the polyhedra where every face consists of parallel pairs of edges. In this case an edge-face duple is compensated by the parallel edge in the same face. Polyhedra of this form are called *generalized zonohedra* [12] . See Figure 1(d). (The term "zonohedron", though originally defined as above by the Russian crystallographer Fedorov, was evolved by Coxeter [4] to mean the more special case where the faces are equilateral; see [12] for the history). For more information on zonohedra, see the web pages [6, 7, 8].

Zonohedra have the property that every face has a parallel face with corresponding edges parallel.

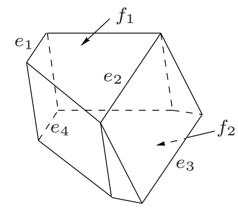

Figure 2: Examples of some compensating edge-face duples: (e_1, f_1) is compensated by (e_2, f_1) and by (e_4, f_2) but not by (e_3, f_2). This is also an equiprojective polyhedron which is not face-compensating: the bottom face f_2 includes two edges (the short ones) that compensate each other, but the remaining edges are compensated by corresponding parallel edges in the top face f_1.

Thus each edge-face duple could alternatively be compensated by the corresponding edge in the parallel face.

More generally we obtain the class of "face-compensating polyhedra", where any face not composed of parallel pairs of edges has a parallel face with corresponding edges parallel. The prisms based on odd regular polygons are in this class, but are not zonohedra.

Finally, there are equiprojective polyhedra that are not face-compensating, for example the one shown in Figure 2.

3 Proof of characterization

Let P be a [convex] polyhedron. Given a direction d, we can distinguish faces of P parallel to d, faces *visible* from d, and faces *invisible* from d. If there are no faces parallel to d, then the edges of the shadow of P projected in direction d are in one-to-one correspondence with the edges of P common to a visible and an invisible face of P. For direction d, let $\mathcal{S}_d$ be the set of edges of P that form edges of the shadow. As d changes continuously, $\mathcal{S}_d$ changes only when faces become parallel to d, on their way between visibility and invisibility or vice versa.

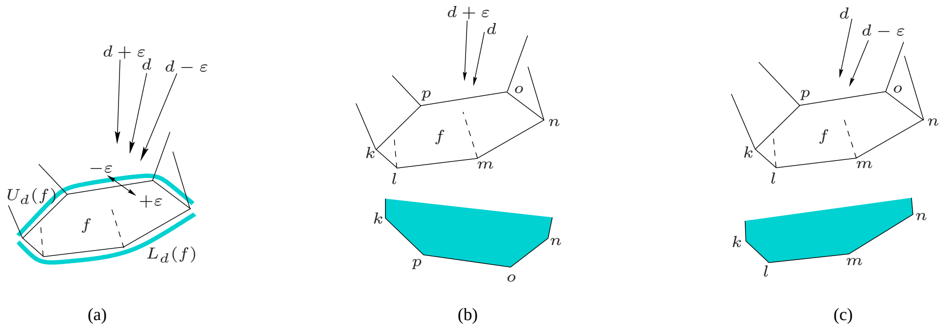

Figure 3: (a) Illustration of ε, and $U_d(f)$ and $L_d(f)$ for a face f; (b) $\mathcal{S}_{d+\varepsilon}(f) = U_d(f)$; (c) $\mathcal{S}_{d-\varepsilon}(f) = L_d(f)$.

Note that if two faces of P are parallel to each other then they both become parallel to d at the same time. Our starting point is the claim that apart from such parallel faces, we can concentrate on the case where d crosses the plane of at most one face at a time.

Lemma 3.1 *We can change d continuously from any initial direction d_s to any other direction d_t, so that for any direction d along the way, the set of faces parallel to d is empty, or consists of one face—and its parallel counterpart if there is one. Furthermore, we can ensure that d crosses the plane of each face orthogonally in a small enough neighbourhood.*

Proof. See full version. □

Thus to show that a polyhedron is equiprojective, it suffices to consider the changes in $\mathcal{S}_d$ as d orthogonally crosses the plane of one face f—and its parallel counterpart f' if it exits—causing f to become visible or invisible. Let $\mathcal{S}_d(f)$ be the edges of f that form edges of the shadow of P in direction d. We will use the notation $\mathcal{S}_d(f, f')$ to mean $\mathcal{S}_d(f)$ together with $\mathcal{S}_d(f')$ if f' exists—i.e. the edges of f and f' that are edges of the shadow.

Take a direction d in the plane of face f, but not in the plane of any other face (except f' if it exists) and let ε be a small vector normal to the plane of f, and consider the directions $d{+}\varepsilon$ and $d{-}\varepsilon$. These two directions make f invisible and visible, respectively, and affect no other faces except f' if it exists. We want to show that $\mathcal{S}_{d+\varepsilon}(f, f')$ and $\mathcal{S}_{d-\varepsilon}(f, f')$ have the same cardinality.

Given a direction d in the plane of face f, let $L_d(f)$ be the lower chain of f with respect to direction d and let $U_d(f)$ be the upper chain. See Figure 3(a). This distinction between upper and lower has to do with visibility in the plane of face f—in particular, $U_d(f)$ consists of the edges of f visible from direction d.

Lemma 3.2 *Let d and ε be as above. Then $\mathcal{S}_{d+\varepsilon}(f) = U_d(f)$ and $\mathcal{S}_{d-\varepsilon}(f) = L_d(f)$.*

Proof. See Figure 3(b, c). □

Corollary 3.1 *If f has a parallel face f' then $\mathcal{S}_{d+\varepsilon}(f') = L_d(f')$ and $\mathcal{S}_{d-\varepsilon}(f') = U_d(f')$.*

The above results give us the machinery we need to prove the sufficiency of our condition for equiprojectivity.

Proof of Theorem 2.1

Sketch only; see full version for detail.
($\Longleftarrow$) For a polyhedron P, if edge-face duples are partitioned into compensating pairs, we can prove that for a direction d in the plane of a single face f [and its parallel counterpart f', if it exists]:

$$|U_d(f)| + |L_d(f')| = |L_d(f)| + |U_d(f')|.$$

Lemma 3.2 and Corollary 3.1 imply that $\mathcal{S}_d$ maintains its cardinality as d orthogonally crosses the plane of one face f [and f']. Then by Lemma 3.1 P is equiprojective.
($\Longrightarrow$) For a polyhedron P, if edge-face duples can't be partitioned into compensating pairs, we find two projections of P of different sizes.

Consider the graph of compensating edge-face duples, which has a vertex for each edge-face duple, and an edge when two duples compensate each other. The parallel family of (e, f) may consist of: (1) one node; (2) two isolated nodes; (3) two nodes joined by an edge; (4) three nodes joined in a path; (5) four nodes joined in a cycle. See Figure 4.

In cases (3) and (5) the parallel family of (e, f) partitions into compensating pairs. In cases (1), (2), and (4) there is no partition into compensating pairs, and we must show that P is not equiprojective. We show it by finding a direction d in the plane

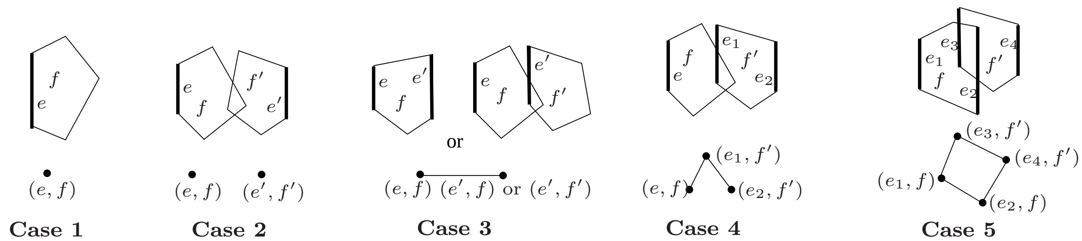

Figure 4: Graph of compensating edge-face duples within one parallel family. Faces f and f' are parallel. Edges drawn in bold are parallel.

of f [and its parallel counterpart f', if it exists] but not in the plane of any other face so that:

$$|U_d(f)| + |L_d(f')| \neq |L_d(f)| + |U_d(f')|.$$

Then from Lemma 3.2 and Corollary 3.1 directions $d + \varepsilon$ and $d - \varepsilon$ yield two projections of different sizes, where ε is a vector perpendicular to f and is small enough to avoid the plane of any other face. □

4 Algorithm

Our characterization provides an $O(n \log n)$ time algorithm to test if a polyhedron is equiprojective. There are $O(n)$ edge-face duples, which we sort by direction and face normal, thereby partitioning the set of edge-face duples into parallel families. Since each family has at most 4 members, it is then a trivial matter to see if it can be partitioned into compensating pairs. See Figure 4.

5 Conclusion

We leave open the question of an algorithm to generate all equiprojective polyhedra, or even to generate just the face-compensating ones. Note that there are algorithms to generate zonohedra [7]. Our most interesting example, the non-face compensating polyhedron in Figure 2, is formed by adjoining two prisms. Can all equiprojective polyhedra be constructed in some way from zonohedra and other face-compensating polyhedra?

We also leave open the question of a linear time algorithm to test equiprojectivity.

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