Partitioning Regular Polygons into Circular Pieces I: Convex Partitions*

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Abstract

We explore an instance of the question of partitioning a polygon into pieces, each of which is as "circular" as possible, in the sense of having an aspect ratio close to 1. The *aspect ratio* of a polygon is the ratio of the diameters of the smallest circumscribing circle to the largest inscribed disk. The problem is rich even for partitioning regular polygons into convex pieces, the focus of this paper. We show that the optimal (most circular) partition for an equilateral triangle has an infinite number of pieces, with the lower bound approachable to any accuracy desired by a particular finite partition. For pentagons and all regular k-gons, $k > 5$, the unpartitioned polygon is already optimal. The square presents an interesting intermediate case. Here the one-piece partition is not optimal, but nor is the trivial lower bound approachable. We narrow the optimal ratio to an aspect-ratio gap of 0.01082 with several somewhat intricate partitions.

1 Introduction

At the open-problem session of the 14th Canadian Conference on Computational Geometry,[1] the first author posed the question of finding a polynomial-time algorithm for partitioning a polygon into pieces, each with an aspect ratio no more than a given $\gamma > 1$ [DO03b]. The *aspect ratio* of a polygon P is the ratio of the diameters of the smallest circumscribing circle to the largest inscribed *indisk*. (We will use "circumcircle" and "indisk" to emphasize that the former may overlap but the latter cannot.) If the pieces of the partition must have their vertices chosen among P's vertices, i.e, if "Steiner points" are disallowed, then a polynomial-time algorithm is known [Dam02]. Here we explore the question without this restriction, but with two other restrictions: the pieces are all convex, and the polygon P is a regular k-gon. Although the latter may seem highly specialized, in fact many of the issues for partitioning an arbitrary polygon arise already with regular polygons. The specialization to convex pieces is both natural, and most in concert with the applications mentioned below. Partitions employing nonconvex pieces will be explored in [DO03a]. Our emphasis in this paper is not on algorithms, but on the partitions themselves.

1.1 Notation

A *partition* of a polygon P is a collection of polygonal *pieces* $P_1, P_2, \ldots$ such that $P = \cup_i P_i$ and no pair of pieces share an interior point. We will use γ for the aspect ratio, modified by subscripts and superscripts as appropriate. $\gamma_1(P)$ is the one-piece γ: the ratio of the radius of the smallest circumcircle of P, to the radius of the largest disk inscribed in P. $\gamma(P)$ is the maximum of all the $\gamma_1(P_i)$ for all pieces P_i in a partition of P; so this is dependent upon the particular partition under discussion; $\gamma_\theta(P)$ is the "one-angle lower bound", a lower bound derived from one angle of the polygon, ignoring all else. This presents a trivial lower bound on any partition's aspect ratio. $\gamma^*(P)$ is the minimum $\gamma(P)$ over all convex partitions of P. Our goal is to find $\gamma^*(P)$ for the regular k-gons. Both the partition and the argument "(P)" will often be dropped when clear from the context.

1.2 Table of Results

Our results are summarized in Table 1. It makes sense that regular polygons for large k are as circular as one could get. We prove that this holds true for all $k \geq 5$. For an equilateral triangle, the optimal partition can approach but never achieve the lower bound $\gamma_{60^\circ} = 3/2$, for finite partitions. The square is an interesting intermediate case. Here the one-piece partition is not optimal, but nor is the trivial lower bound approachable. So $\gamma^* \in (\gamma_{90^\circ}, \gamma_1)$. We narrow the optimal ratio to a

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small gap by raising the lower bound and lowering the upper bound.

Regular Polygon	γ_1	γ_θ	γ^*	k^*
Triangle	2.00000	1.50000	γ_θ	∞
Square	1.41421	1.20711	$\in J$	∞?
Pentagon	1.23607	1.11803	γ_1	1
Hexagon	1.15470	1.07735	γ_1	1
Heptagon	1.10992	1.05496	γ_1	1
Octagon	1.08239	1.04120	γ_1	1
k-gon	$\frac{1}{\cos(\pi/k)}$	$\frac{1+\csc(\theta/2)}{2}$		

Table 1: Table of Results on Regular Polygons. γ_1: one-piece partition; γ_θ: single-angle lower bound; θ: angle at corner; γ^*: optimal partition; k^*: number of pieces in optimal partition; $J = [1.28868, 1.29950]$.

1.3 Motivation

Our motivation for investigating circular partitions is for their advantages in several application areas. For instance, circular polygons can be tightly circumscribed by circles and therefore support quick collision detection tests.

There is an interesting connection between the problem studied in this paper and packings and coverings. For any partition of P, the collection of indisks for each piece of the partition forms a packing of P by disks, and the collection of circumcircles enclosing each piece form a covering of P. We have found this connection to packing and covering more relevant when the pieces are not restricted to be convex [DO03a], but even for convex pieces it is at least suggestive. For example, the notorious difficulty of packing equal disks in a square may relate to the apparent difficulty of our problem for the square, which leads to a packing of the square with unequal disks. In any case, despite the connections, we have not found in the literature any work that directly addresses our particular problem.

1.4 Outline of Paper

Rather than follow the ordering in the table, we proceed in order of increasing difficulty: starting with the easiest result (pentagons and $k > 5$, Section 2), moving next to the equilateral triangle (Section 3), and finally to the square (Section 4). We look to the natural next steps for this work in a final Discussion, Section 5. Proofs of theorems and lemmas are omitted due to space constraints.

2 Pentagon

It is clear that for large enough k, $\gamma^* = \gamma_1$ for a k-gon. The only question is for which k does this effect take over. The answer is $k = 5$:

Theorem 1 *For a regular pentagon ($k = 5$), the optimal convex partition is just the pentagon itself, i.e., $\gamma^* = \gamma_1$ and $k^* = 1$.*

3 Equilateral Triangle

Recall from Table 1 that the one-angle lower bound for the equilateral triangle is $\gamma = 3/2$. In this section we show that this lower bound can be approached, but never achieved (a general result), by a finite convex partition:

Lemma 2 *For any polygon P, the one-angle lower bound $\gamma_\theta(P)$ can never be achieved by a finite convex partition.*

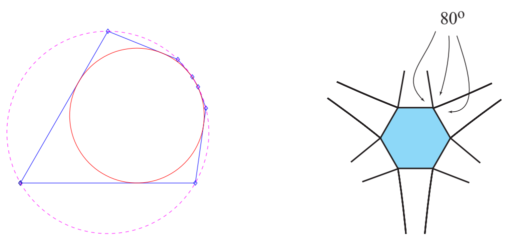
Define an $80°$-*quadrilateral* as one whose corner angles fall within $90° \pm 11°$. Define a *pair of* $80°$-*curves* $A(t)$ and $B(t)$ as two curves parametrized by $t \in [0, 1]$, that satisfy the property that each segment $C(t)$ whose endpoints are $A(t)$ and $B(t)$, meets the curves at angles in the range $90° \pm 11°$.

Our proof that the lower bound of $\gamma = 3/2$ can be approached for an equilateral triangle follows five steps:

1. $\gamma = 3/2$ suffices to partition any rectangle .
2. $\gamma = 3/2$ suffices to partition any "$80°$-quadrilateral" .
3. $\gamma = 3/2$ suffices to approximately partition the region between any pair of "$80°$-curves" .
4. The corners of an equilateral triangle can be covered by a polygon with γ_1 as near to $3/2$ as desired (Figure 1a).
5. The remaining "interstice" can be partitioned into regions bound by $80°$-curves (Figure 1b).

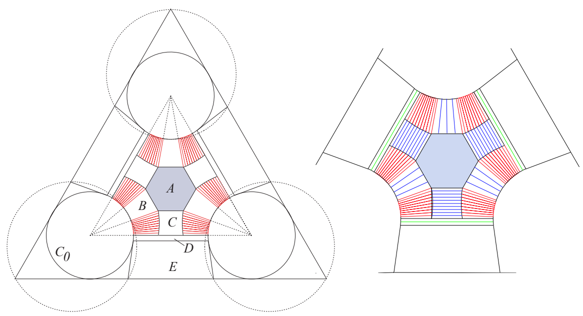
The overall design of the full partition is illustrated in Figure 2a. This figure shows several parts of the construction, but is not a full partition. The region B is easily partitioned into $80°$-quadrilaterals by more radial rays from the center of C_0. How many rays depends on the polygonal approximation to the corner piece; see Figure 1a. The region C can be partitioned by horizontal lines; again all the quadrilaterals are $80°$-quadrilaterals .

The $80°$-curves terminate on the line through the centers of the two bottom incirles. Piece D of the partition would again be partitioned by horizontal segments, as many as are needed to accommodate the polygonal approximation to C_0 there. Finally, the large remaining quadrilateral E has base angles $> 82°$, and so is already an $80°$-quadrilateral. A possible partition into $80°$-quadrilaterals is shown in Figure 2b.



(a) (b)

Figure 1: (a) A heptagon covering a $60°$-corner, with $\gamma = 1.5012$. (b) The $120°$ hexagon angle meets three $80°$ angles.



(a) (b)

Figure 2: (a) Overall design of partition of equilateral triangle (b) Partition into $80°$-quadrilaterals.

Theorem 3 *An equilateral triangle may be partitioned into a finite number of pieces with ratio γ, for any $\gamma > 3/2$. As γ approaches $3/2$, the number of pieces goes to infinity.*

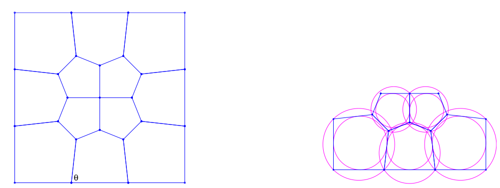
4 Square

The square is intermediate between the equilateral triangle and the regular pentagon in several senses: (a) Unlike the pentagon, the one-piece partition $\gamma_1 = 1.41421$ is not optimal; (b) Unlike the equilateral triangle, the one-angle lower bound of $\gamma_{90°} = 1.20711$ (Table 1) cannot be approached. Although we believe a result similar to Theorem 3 holds—there is a lower bound that can be approached but not reached for a finite partition—we have only confined the optimal ratio γ^* to the range $[1.28868, 1.29950]$, leaving a gap of 0.01082.

We first show in Sections 4.1 and 4.2 that the one-piece partition is not optimal through a series of increasingly complex partitions. Then we establish an upper bound $\gamma^* \leq 1.29550$ and close with a conjecture in Section 4.3.

4.1 Pentagons on Side

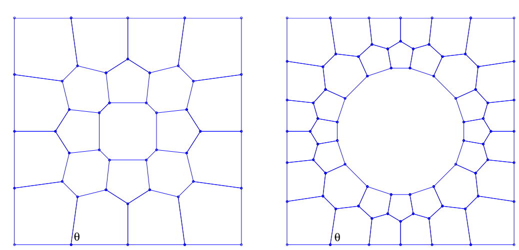
To improve upon the one-piece partition, pieces that have five or more sides must be employed, for the square itself is the most circular quadrilateral. One quickly discovers that covering the side of the square is challenging and crucial to the structure of the overall partition. We first explore partitions that use pentagons around the square boundary. Figure 3 shows a 12-piece partition that already improves upon $\gamma_1 = 1.41421$, achieving $\gamma = 1.33964$; but it is easily seen to be suboptimal. Our next attempt, Figure 4a, is a 21-piece partition



(a) (b)

Figure 3: 12-piece partition achieving $\gamma = 1.33964$ (a) Overall design. Here $\theta = 89.62°$. (b) Details of critical indisks and circumcircles.

with a central octagon, improving to $\gamma = 1.32348$. Our most elaborate pentagon example is the 37-



(a) (b)

Figure 4: (a) 21-piece partition achieving $\gamma = 1.32348$. Here $\theta = 82.16°$. (b) 37-piece partition achieving $\gamma = 1.31539$. Here $\theta = 81.41°$.

piece partition shown in Figure 4b, which achieves $\gamma = 1.31539$. Its center is a near-regular 16-gon. We will leave it as a claim without proof that any square partition that covers the entire boundary of the square with pentagons must have $\gamma \geq 1.31408$. So Figure 4b cannot be much improved.

4.2 Hexagons on Side

To make further advances, it is necessary to move beyond pentagons. Figure 5 shows our best partition, which employs four corner pentagons and four hexagons along each side of the square. As is apparent from the figure, it is no longer straightforward to fill the interior after covering the boundary. The interior includes four heptagons adjacent to the corner pieces, and four dodecagons at the center, with all remaining pieces hexagons or pentagons.

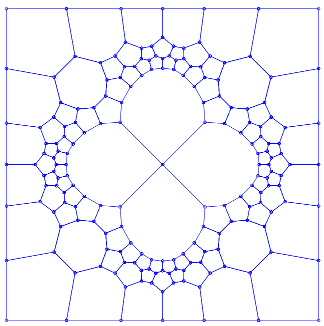


Figure 5: 92-piece partition achieving $\gamma = 1.29950$

We again leave it as claim without proof that further improvements here will not be large: any square partition that covers the entire boundary of the square with hexagons must have $\gamma \geq 1.29625$.

4.3 Lower Bound on γ^*

Note that throughout our series of partitions, the angle θ on the square side between the corner piece and its immediate neighbor is a critical angle. One wants this small so that the corner piece can be circular; but too small and the adjacent piece cannot be circular. It is exactly this tension that we exploit to establish a lower bound on γ^*:

Theorem 4 *The optimal aspect ratio for a convex partition of a square is at least*

$$\gamma^* \geq 1.28868$$

Based on partial results not reported in this paper, we would be surprised if the optimal partition could be achieved with a finite partition:

Conjecture 1 *No finite partition achieves the optimal partition of the square: rather γ^* can be approached as closely as desired as the number of pieces goes to infinity.*

5 Discussion

This paper establishes the optimally circular convex partition of all regular polygons, except for the square, where we have left a small gap. We hope to show in future work that our results apply to arbitrary polygons as well. We have also investigated nonconvex circular partitions of regular polygons [DO03a].

Our work leaves many problems unresolved:

1. Narrow the gap $[1.28868, 1.29950]$ for the optimal aspect ratio of a convex partition of a square.
2. Determine for the square if k^*, the number of pieces in an optimal partition, is finite or infinite (cf. Conjecture 1).
3. For each k, find the optimal ratio $\gamma^{(k)}$ of a polygon P using only convex ($\leq k$)-gons. It is especially interesting to determine $\gamma^{(k+1)}$ for a k-gon. For example, Figure 3 and work not reported here shows that $\gamma^{(5)} \in [1.31408, 1.33964]$ for a square.
4. For each k, find the optimal ratio γ_k of a polygon P using $\leq k$ convex pieces. So each of our partitions of the square establishes a particular upper bound, e.g., Figure 4 shows that $\gamma_{21} \leq 1.32348$.
5. More generally, develop a tradeoff between the number of pieces of the partition and the circularity ratio achieved.
6. Extend our results to all convex polygons, and then to arbitrary polygons.

Finally, all these problems could be fruitfully explored in 3D, the natural dimension for the applications mentioned in Section 1.3.

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