# Parallel Morphing of Trees and Cycles*

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## Abstract

We prove that for any two simple chains [more generally, trees] in $R^d$ with corresponding edges parallel, there is a *parallel morph* between them—i.e. a morph in which all intermediate chains [trees] remain simple and parallel to the original. A similar result had been proved by Guibas et al. [8] for simple cycles in $R^2$. We prove that the result for cycles does not extend to $R^3$ by giving two simple cycles, with corresponding edges parallel, that represent the same knot, and yet have no parallel morph.


## 1 Introduction

A *morph* is a continuous transformation, or "metamorphosis" of one image or object to another. Morphs can be used to create visual effects, simulate growth, etc., and are of great interest in computer graphics for these reasons. See [6]. This paper is concerned, not with visual effects, but with maintaining the geometric structure of objects throughout a morph; specifically, we wish to morph one configuration of line segments to another while preserving incidence relationships and the directions of the segments. This problem bears some similarity to linkage reconfiguration problems, where we wish to morph one configuration of line segments to another while preserving incidence relationships and the *lengths* of the segments.

Given a graph $(V, E)$ where where $V$ is a set of vertices and $E$ is a set of edges of the form $(u, v)$, $u, v \in V$, a *straight-line drawing* of $(V, E)$ is a triple $(V, E, p)$ such that $p$ is a mapping from each $v \in V$ to a unique point in $d$-dimensional Euclidean space. Although, in general, drawings of graphs need not have straight-line edges, throughout this paper we discuss only straight-line drawings and refer to these simply as "drawings". Further, we are only interested in drawings that are *simple*, in that edges of the drawing are pairwise disjoint, except when they meet at a common vertex. In the special case that every edge of a drawing is oriented parallel with one of the $d$ axes, we say that the drawing is *orthogonal*.

Two drawings of the same graph, $P = (V, E, p_P)$ and $Q = (V, E, p_Q)$, are said to be *parallel* if for every edge $(u, v) \in E$, its drawing in $P$ is parallel with its drawing in $Q$. A *parallel morph* between parallel drawings $P, Q$ of the same graph is a continuously changing family of drawings, $R(t), 0 \leq t \leq 1$, with $R(0) = P$ and $R(1) = Q$, such that each $R(t)$ is parallel with $P$ and $Q$. Moreover, we require that every drawing $R(t)$ is simple, that is, no two edges may meet except at a common vertex.

This paper addresses the question: Given two parallel drawings of a graph, do they admit a parallel morph? We concentrate on restricted graph classes, namely paths, trees and cycles. Paths are a special case of trees but can be treated more simply. To summarize our results: Parallel drawings of paths and trees always admit a parallel morph in any $d$-dimensional space. However, there exist parallel drawings of cycles in 3-dimensional space that do not admit a parallel morph, even though the cycles form the same knot.

In the plane, simple drawings of cycles are simple polygons. In work that motivated this research, Guibas et al. [8] and Grenander et al. [7] show that simple parallel polygons always admit a parallel morph. (Also see [10] for the case of orthogonal polygons.) Parallel morphing of trees in the plane follows from this result, since a tree can be converted to a polygon by "thickening" each edge. However, this approach does not apply in higher dimensions.

Two polygons are parallel iff they have the same sequence of internal angles and the same first edge direction. Vijayan [11] proved that a given angle sequence is realized by some simple polygon iff the obvious angle sum condition holds. Any realizable angle sequence can be realized by a family of parallel polygons. The question of whether two simple parallel polygons have a parallel morph can be viewed as a question of connectivity within this family. Realizability of angles for more general plane graphs is NP-complete [5]. In three dimensions the question of whether a sequence of angles is realized by a simple polygon is solved for the orthogonal case [1].

We conclude this section by discussing the connections between parallel morphing and linkage reconfiguration problems. To describe this connection, we need to mention parallel redrawings and rigidity theory. A drawing of a graph has a *parallel redrawing* [12] if the vertices can be moved such that all edges remain parallel to those in the original drawing, and the resultant drawing is neither a translation nor a scaling of the original.

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Note that maintaining the simplicity of the drawing is not an issue. Whiteley [12], and Servatius and Whiteley [9] study questions of the *existence* of a parallel redrawing. This turns out to be directly related to questions in rigidity theory.

In rigidity theory, frameworks are composed of rigid bars, idealized as straight-line edges and attached at vertices. Where two bars meet at a vertex, they are allowed to move freely with respect to each other. The fundamental problem is that of deciding whether or not a framework is *rigid*. That is, is there a non-trivial *infinitesimal motion* that moves the vertices while keeping the lengths of the bars fixed. Simplicity is not an issue. To contrast the two situations: in parallel redrawings edge lengths may change but angles are fixed, whereas in rigidity theory, edge lengths are fixed and angles may change. The answers are the same however: a configuration has a parallel redrawing iff it is not rigid, since the vectors orthogonal to an *infinitesimal motion* provide a parallel redrawing. See [9].

Linkage reconfiguration problems also deal with rigid bars and flexible angles, however—in contrast with rigidity theory—simplicity must be maintained, and the questions are about reachability: given two structures that are composed of the same set of rigid bars and with the same combinatorial structure, can we morph from one to the other preserving incidence relationships and the lengths of the bars? Linkage reconfiguration has received much attention from the computational geometry community in recent years. Chains and polygons in 2D can be reconfigured [4], but chains in 3D cannot [2].

Thus linkage reconfiguration problems are to rigidity theory as parallel morphs are to parallel redrawings. The fact that rigidity is equivalent to the non-existence of parallel redrawings opens the door to the possibility that techniques from linkage reconfiguration may apply to parallel morphs. However, there is no simple correspondence, since, as we show in this paper, the answers are different: chains in 3D have parallel morphs, but cannot always be reconfigured.

## 2 Morphing Paths

Given a path $(V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{(v_1, v_2), (v_2, v_3) \ldots, (v_{n-1}, v_n)\}$, let $P = (V, E, p_P)$ and $Q = (V, E, p_Q)$ be parallel, simple drawings of $(V, E)$ in $R^d$. We show that such drawings always admit a parallel morph.

We simplify the problem by defining a *canonical form* for each of $P$ and $Q$ such that the canonical forms of both $P$ and $Q$ are identical up to a scaling operation. Thus, a parallel morph from $P$ to $Q$ can be accomplished by morphing $P$ to its canonical form, scaling the entire drawing, and then reversing the morph between $Q$ and its canonical form.

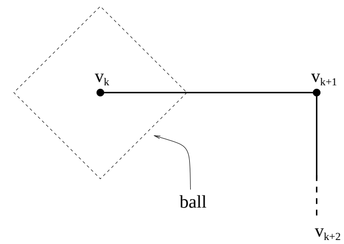


Figure 1: The ball, after morphing edges $(v_1, v_2), \ldots, (v_{k-1}, v_k)$.

### 2.1 Orthogonal drawings

In this subsection, we limit our discussion to orthogonal drawings. Define the *minimum feature size* $\delta_P$ of $P$ to be the minimum of (1) the smallest distance between non-incident edges of $P$, and (2) the length of the shortest edge in $P$. Define the *canonical form* of $P$ to be the parallel drawing of $P$ in which each edge $(v_i, v_{i+1})$ has length $\delta_P 2^{i-n-1}$. Note that any two parallel drawings $P, Q$ have the same canonical form up to a scaling operation, where the size of this scaling depends upon the ratio between $\delta_P$ and $\delta_Q$.

The morph works as follows: For $i = 1$ to $n-1$, shrink edge $(v_i, v_{i+1})$ to length $\delta_P 2^{i-n-1}$. Clearly, following this morph we have a drawing that is the canonical form of $P$, and every intermediate drawing of the morph is parallel to $P$.

**Claim 1** *Every intermediate drawing of the morph (including the canonical drawing) is simple.*

**Proof:** By induction. Notice that after shrinking edge $(v_{k-1}, v_k)$, all preceding edges lie entirely within a $d$-dimensional ball centered at $v_k$—as illustrated in Figure 1—whose $L_1$ (rectilinear) radius is equal to the sum of the lengths of the edges $(v_1, v_2)$ to $(v_{k-1}, v_k)$:

$$\delta_P \sum_{i=1}^{k-1} 2^{i-n-1} = \delta_P 2^{-n}(2^{k-1} - 1) < \delta_P 2^{k-n-1} < \delta_P$$

When edge $(v_k, v_{k+1})$ shrinks, the edges inside the ball remain static with respect to each other. Likewise, the edges outside of the ball remain static respect to each other. We need only show that the edges inside the ball cannot intersect those outside the ball.

Because the final length of $(v_k, v_{k+1})$ is greater than the radius of the ball, shrinking $(v_k, v_{k+1})$ leaves the ball disjoint from $v_{k+1}$ and from its next incident edge if there is one. No other edge outside the ball can enter the ball since they all have distance at least $\delta_P$ from the edge $(v_k, v_{k+1})$, and $\delta_P$ is greater than the radius of the ball. $\square$

**Theorem 1** *Any two parallel, orthogonal drawings $P, Q$ of a path $(V, E)$ embedded in $d$-dimensional space admit a parallel morph.*

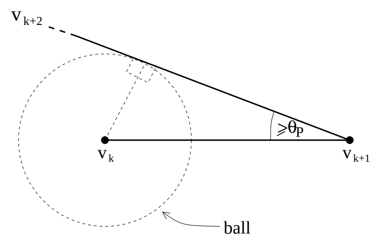


Figure 2: The $d$-dimensional ball after morphing the first $k-1$ edges.

### 2.2 General drawings of paths

To generalize the above result to non-orthogonal drawings of paths, we redefine the canonical form of a drawing. As before, let $\delta_P$ denote the minimum feature size of a drawing $P$. Let $\theta_P$ denote the size of the the minimum angle between two incident edges of $P$. Define, $\Gamma = \frac{2}{\sin(\min\{\pi,\theta_P\})}$.

We now define the canonical form of $P$ as the drawing that is parallel to $P$ in which each edge $(v_i, v_{i+1})$ has length $\delta_P \Gamma^{i-n-1}$.

Since two parallel drawings $P$ and $Q$ necessarily share the same minimum angle, the canonical drawings of $P$ and $Q$ are identical up to a scaling operation. Every edge in the canonical drawing of $P$ is of length at most $\delta_P \Gamma^{-2}$, and is hence smaller than the corresponding edge in $P$. As in the orthogonal case, to morph $P$ to its canonical form we shrink each edge in turn to its final length. Thus, every intermediate drawing of the morph is parallel to $P$.

**Claim 2** *Every intermediate drawing of the morph (including the canonical drawing) is simple.*

**Proof:** By induction. Assume that the morphing procedure maintains simplicity throughout the shrinking of edges $(v_1, v_2)$ to $(v_{k-1}, v_k)$. As illustrated in Figure 2, these edges reside within a $d$-dimensional ball centered at $v_k$ whose $L_2$ (Euclidean) radius is

$$\delta_P \sum_{i=1}^{k-1} \Gamma^{i-n-1} = \delta_P \Gamma^{-n} \left( \frac{1-\Gamma^{k-1}}{1-\Gamma} \right)$$

By a similar argument as used in the proof of Claim 1, the radius of the ball never exceeds $\delta_P$. Thus, as $(v_k, v_{k+1})$ shrinks the ball centered at $v_k$ cannot intersect any edge outside of the ball that is not incident upon $(v_k, v_{k+1})$. On the other hand, if $\pi \leq \delta_P$, then the arguments of Claim 1 apply to show that the ball does not intersect an the next edge incident with $v_{k+1}$.

So, assume that $\pi > \delta_P$. As $(v_k, v_{k+1})$ shrinks, $v_k$ will get continually closer to edge $(v_{k+1}, v_{k+2})$. When the edge has been shrunk to its final length, the distance from $v_k$ to $(v_{k+1}, v_{k+2})$ will be at least $\sin\theta_P(\delta_P \Gamma^{k-n-1})$. It is not difficult to show that this distance is greater than the radius of the ball centered at $v_k$, thus the ball cannot intersect $(v_{k+1}, v_{k+2})$. □

**Theorem 2** *Any pair of parallel drawings of a path embedded in a $d$-dimensional space admit a parallel morph.*

## 3 Trees

Let $P$ be a drawing of a tree, such that $\delta_P$ is the minimum feature size, and $\theta_P$ is the minimum angle. We root the tree at one vertex and let $d(v)$ be the distance from vertex $v$ to the root.

We define the canonical form of $P$ as the drawing that is parallel to $P$, such that for each edge $(u, v)$ where $d(u) = d(v) + 1$, the length of the edge is

$$\delta_P \left( \frac{2}{\sin(\min\{\pi, \theta_P\}/2)} \right)^{d(u)-n-1}$$

We morph by shrinking each edge to its canonical length, taking the edges in decreasing order of their distance from the root. Using a proof similar to that of Theorem 2 we obtain:

**Theorem 3** *Parallel drawings of a tree in $d$-dimensional space always admit a parallel morph.*

## 4 Cycles

Clearly, parallel cycles in $R^3$ that form different knots do not have parallel morphs. More interesting is:

**Theorem 4** *Not all orthogonal, parallel drawings of cycles admit a parallel morph, even if they represent the same knot.*

**Proof:** We prove this for the cycles in Figure 3. Consider the possible orderings of the vertices along each of the three axes. In fact, only one such ordering is possible with respect to the $x$ and $z$ axes. For each vertex $v \in \{a, \ldots, j\}$, let $v_x$, $v_y$ and $v_z$ denote the $x$, $y$ and $z$ coordinates of $v$, respectively.

With respect to $x$ coordinates, two vertices connected by an edge parallel to the $y$ or $z$ axis have equal $x$ coordinates, and two vertices connected by an edge parallel to the $x$ axis have a fixed order. Thus the $x$ ordering of the vertices is completely determined:

$$c_x = d_x = e_x < f_x = g_x = h_x = i_x < j_x = a_x = b_x$$

Likewise, with respect to $z$ coordinates:

$$h_z = i_z = j_z = a_z < b_z = c_z < d_z = e_z = f_z = g_z$$

Consider the edges $(b, c)$ and $(g, h)$. By the above equations, $c_x < g_x = h_x < b_x$ and $h_z < b_z = c_z < g_z$ at

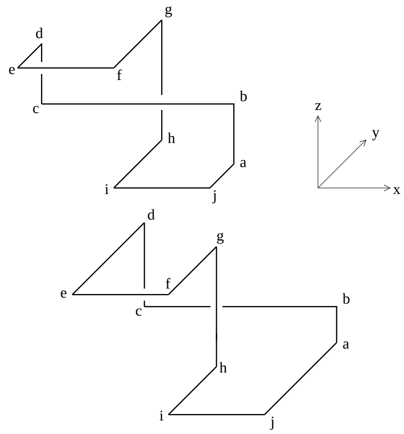


Figure 3: Two drawings of a cycle that do not admit a parallel morph.

all times throughout a parallel morph. Note that $(b, c)$ and $(g, h)$ intersect with respect to $x$ and $z$ coordinates. If, in addition, $b_y = c_y = g_y = h_y$, then the edges intersect, which is not allowed in a simple parallel morph. In the uppermost drawing of Figure 3, $b_y = c_y < g_y = h_y$, while in the lowermost drawing we have the reverse, $b_y = c_y > g_y = h_y$. However, to get from one drawing to the other there must be an intermediate drawing in which the $y$-coordinates of both edges are equal. Thus, there exists no parallel morph from the upper to the lower drawing of Figure 3. $\square$

## 5 Open Problems

• What is the complexity of deciding whether cycles in 3D admit a parallel morph?
• Do [edge sets of] polyhedra in 3D admit parallel morphs?
• Are there parallel morphs of paths/trees that keep edge lengths well-behaved? See [3] for results on cycles.

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