Curves of Width One and the River Shore Problem

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Abstract

We consider the problem of finding the shortest curve in the plane that has unit width. This problem was first posed as the "river shore" puzzle by Ogilvy (1972) and is related to the area of on-line searching. Adhikari and Pitman (1989) proved that the optimal solution has length 2.2782... We present a simpler proof, which exploits the fact that the width of a polygon does not decrease under a certain convexification operation.

1 Introduction

The competitive analysis of robot navigation techniques requires various search primitives that can be readily used in the solution of more complex problems. For example, two-ray searching and its m-ray generalization [5] form the base of various search algorithms (e.g., [3, 4]). While studying various three-dimensional search problems (work in progress), the need for one such primitive arose in the form of the question of what was the shape and length of the shortest curve of width one.

Definition 1 *Let Γ be a path. A* supporting line *of Γ is a line L that intersects Γ in at least one point and such that Γ is entirely contained in one of the two closed halfplanes defined by L.*

Definition 2 *The* width *of a curve is defined as the minimum distance between any two distinct parallel lines supporting the curve.*

Interestingly, it turns out that the question of what is the shortest curve of width one has been asked before in the setting of recreational mathematics [6].

> *The River Shore Problem:* Starting at an unknown point inside a river of width one, what is the shortest path that is guaranteed to reach one of the two shores of the river?

It is not hard to see that this is equivalent to finding the shortest curve in $\mathbb{R}^2$ that has width one. This question was reported as an open problem by Ogilvy [6].

For the closed curve case, the circle of diameter one has width one and has perimeter π, which is optimal. Surprisingly, there are infinitely many closed curves (so-called *curves of constant width*) with the property that the width of the minimum enclosing strip along any direction is one, and all of these curves have the same perimeter π (this is known as *Barbier's theorem*). However, for open curves, shorter solutions are possible. For example, a V-shape formed by the vertices of an equilateral triangle already gives length $4/\sqrt{3} = 2.3094\ldots$

In this paper, we obtain a curve of width one and length 2.2782..., which is optimal.[1] This curve was actually first discovered by Adhikari and Pitman [1] in 1989, although we were unaware of their result when the initial draft of this paper was written. We give a different proof of optimality, though, that is simpler and requires less steps and less cases than Adhikari and Pitman's. Our proof exploits an interesting lemma, stating that a certain convexification procedure used in computational geometry (e.g., by Aloupis et al. in last year's CCCG [2]) can only increase the width of a given polygon or polygonal chain.

2 Upper Bound

We begin by constructing a specific curve of width one and length 2.2782... First, we restrict ourselves to solutions with a generalized V-shape. For convenience, we invert the shape (as in Figure 1) and call it a "yurt"[2]:

Definition 3 *A* yurt *is a curve that starts at the origin s and ends at a point t on the x-axis, such that*

1. *the apex (highest point) v is on or above the line $y = 1$,*
2. *the portion of the curve from v to t encloses the circular arc of radius one centered at s and, symmetrically,*
3. *the portion of the curve from s to v encloses the circular arc of radius one centered at t.*

[1] The curve presented here is introduced visually in the 2003 SoCG video session.
[2] Yurt: A tent used by nomadic peoples of central Asia.

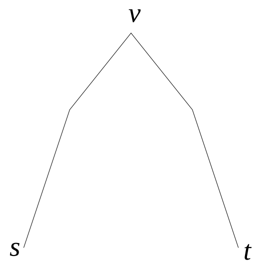

Fig. 1: A yurt

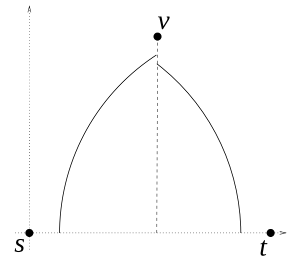

Fig. 2: The circular arcs

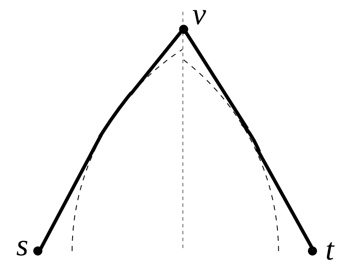

Fig. 3: The upper convex hull

Lemma 1 *Every yurt curve has width at least one.*

Proof. The width of the curve is at least the width of the convex hull H of s, t, v and the two circular arcs. (See Figures 2 and 3.) Any pair of lines supporting H must go through at least one of s, t and v. Because v is on the line $y = 1$ and H contains the circular arcs of radius one, the supporting line on the other side must be at a distance of one or greater. □

The shortest yurt curve is the upper convex hull of s, t, v and the two circular arcs, with v as low as possible (i.e., on the $y = 1$ line). We now determine the best choice of x-coordinates. Let $t = (u, 0)$. We observe that in the optimal solution, the point v is located halfway between s and t, i.e., $v = (u/2, 1)$. This can be seen by a reflection technique commonly used in shortest path computations. We reflect the portion of the curve from v to t using the line $y = 1$ as a mirror (see Figure 4) and obtain a path from s to the reflected point $t' = (u, 2)$ avoiding two circles. The shortest path is through the common tangent of these circles, which intersects $y = 1$ at $v = (u/2, 1)$.

It follows then that the only free parameter is the value u, which uniquely determines the position of b and v and hence the shape of the entire curve. The length of the curve in terms of u is

$$u + 2\sqrt{u^2 - 1} - 2\arccos(1/u) + 2\arccos\Big(4u/(4 + u^2)\Big).$$

We determine the best value for u using calculus and find that the minimum length is $2.2782\ldots$ for $u = 2\sqrt{z} = 1.0434\ldots$, where $z = 0.2722\ldots$ is a root of the cubic $3z^3 + 9z^2 + z - 1$. See Figure 5. This yields the shortest yurt curve.

3 Lower Bound

Our preceding derivation of the shortest yurt curve is similar to Adhikari and Pitman's [1]. To prove optimality, it remains to show that a shortest curve of width one indeed belongs to the yurt family; here, our proof departs from Adhikari and Pitman's and is much shorter.

We establish a series of simple lemmas that progressively restrict the types of shapes that the shortest curve of width one can take.

One property about the optimal curve is that it must be the shortest path through the vertices of its convex hull, and consequently is composed of one or more chains of the boundary of the convex hull joined by noncrossing diagonals. We derive a stronger property: an optimal curve in fact involves just one convex chain. This property seems less obvious (for example, see Adhikari and Pitman's proof [1]). Nonetheless, with the right approach, we show how this property can be proved elegantly. The idea is inspired by a convexification strategy studied by Aloupis et al. [2].

Lemma 2 (Convexification lemma) *Given a (possibly self-intersecting) polygon P with edges oriented in clockwise order, let P' be the polygon formed by arranging the edges of P via translations so that the directions of the edges form a monotonic sequence (as a result, P' is a convex polygon). Then the width of P' is at least the width of P.*

Proof. Let V be the set of all vectors describing the (oriented) edges translated to the origin. Take an arbitrary direction described by a unit normal vector $\vec{d}$. The distance between supporting lines along this direction is the absolute value of the sum $\sum_{\vec{v}_i \in S} \vec{d} \cdot \vec{v}_i$ over some subset $S \subseteq V$. Clearly, this quantity is upperbounded by

$$\sum_{\vec{v}_i \in V:\ \vec{d}\cdot\vec{v}_i > 0} \vec{d} \cdot \vec{v}_i \;=\; -\sum_{\vec{v}_i \in V:\ \vec{d}\cdot\vec{v}_i < 0} \vec{d} \cdot \vec{v}_i.$$

This upper bound is attained when the polygon in question is P', due to the convexity of P'. □

Lemma 3 *There is a shortest curve of width one which is completely contained in the boundary of its convex hull.*

Proof. Let Γ be a shortest curve of width one with endpoints s and t. To ease the argument, imagine that Γ is polygonal (this assumption can be removed by taking a limit). Let $P = \Gamma \cup \{st\}$. Form P' as above. Consider the new curve $\Gamma' = P' - \{st\}$. The length of Γ' is equal to the length of Γ, but the width of Γ' is at least the width of Γ by the convexification lemma. □

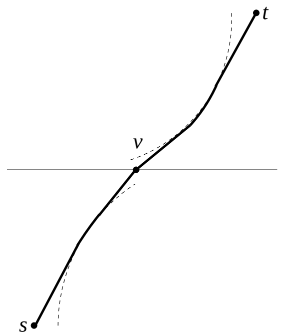

Fig. 4: Mirrored curve

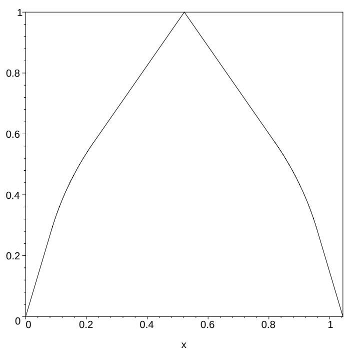

Fig. 5: The optimal yurt

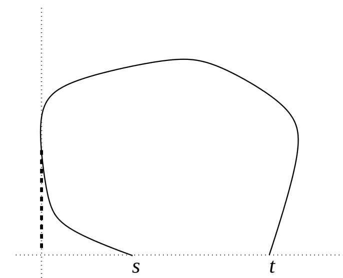

Fig. 6: Vertical supporting line

Having proved the main property, we can easily derive the subsequent lemmas.

Lemma 4 *Some shortest curve of width one has starting and ending point s and t on the x-axis, is supported by the x-axis, and is supported by vertical lines at s and t.*

Proof. The previous lemma implies that st forms a supporting line. By rotation, we may assume that s and t lie on the x-axis. If the vertical line at s is not a supporting line, let w be the first point from s along the curve that defines a vertical supporting line. By removing the portion of the curve from s to w and adding a vertical line segment from the x-axis to w, we get a curve of width at least one and of shorter length (see Figure 6). Thus, there must be a vertical supporting line at s, and similarly at t. □

Lemma 5 *Some shortest curve of width one is a yurt curve.*

Proof. For the curve from the previous lemma, because the x-axis is a supporting line, the apex v must lie on or above the line $y = 1$. By translation, we can make s the origin. Because the curve is contained in the first quadrant, the portion from v to t must enclose the circular arc centered at s and of radius one. Similarly, the portion from s to v must enclose the circular arc centered at t and of radius one. □

Putting everything together, we obtain the main theorem.

Theorem 1 *There is a curve of width one and length* $2.2782\ldots$ *Furthermore, there is no shorter curve of width one.*

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