On Shortest Paths in Line Arrangements

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Abstract

In this paper, we show that the shortest path between two points in a grid-like arrangement of two pencils of lines has a particularly simple structure, as was previously conjectured. This gives a linear-time algorithm for computing shortest paths in such arrangements.

1 Introduction

In this paper, we look at the problem of finding shortest paths in arrangements of lines. Suppose one has an arrangement of lines and wants to compute a shortest path between two given vertices of the arrangement, where the path is restricted to points on the lines. This can be viewed as a city tour problem, where the lines in the arrangement are viewed as roads in the city and we want to find a shortest path between two points in the city, where any path can travel only on the roads.

The problem can be solved by associating a weighted graph to the input set of lines - add a node to the graph corresponding to each vertex of the arrangement, add an arc to the graph corresponding to each edge of the arrangement with the weight of the arc being the length of the corresponding edge. In the worst case, there might be $\Theta(n^2)$ arcs in the graph where n is the number of lines in the arrangement. Klein et al. in [6] have shown how to compute shortest paths in a planar graph in linear time; hence a shortest path in arrangements can be found in $O(n^2)$ time. This is the best known time bound for this problem.

1.1 Previous Results

Eppstein and Hart looked at a special class of arrangements which consisted only of vertical and horizontal *segments* [2] and showed that one can compute a shortest path in $O(n^{1.5})$ time and $O(n^{1.5})$ space. M. van Kreveld then improved this to $O(n \log n)$. Eppstein and Hart in [3] looked at classes of arrangement where there are only k different slopes among the input lines, and they give an $O(n+k^2)$ algorithm to compute a shortest path in such arrangements.

Bose et al. [1] gave the first approximation algorithm for this problem. They gave a 2-approximation algorithm that runs in $O(n \log n)$ time. Hart in [5] gave an algorithm that finds a $1+\epsilon$ approximation of the shortest path in time $O(n \log n + \min(n, \frac{1}{\epsilon^2})\frac{1}{\epsilon} \log \frac{1}{\epsilon})$.

1.2 New Results

Let L be a set of lines (which we call a pencil) that pass through point p and K be a pencil of lines through another point $q \neq p$. Each line in $L \cup K$ is distinct from the line ℓ_{pq} that passes through p and q. Any pair of lines $l \in L$ and $k \in K$ intersect. Moreover, all such intersection points lie on the same side of ℓ_{pq}. Let H_{pq} be the closed half-space defined by ℓ_{pq} that contains all such intersection points in its interior. For any $l \in L$ (resp. $k \in K$), let $\alpha(l)$ (resp. $\alpha(k)$) denote the angle between the ray $l \cap H_{pq}$ (resp. $k \cap H_{pq}$) and the segment pq (resp. qp). Suppose $L = \{l_1, \ldots, l_n\}$ is ordered such that $\alpha(l_i) < \alpha(l_{i+1})$ and $K = \{k_1, \ldots, k_m\}$ is ordered such $\alpha(k_i) < \alpha(k_{i+1})$. Let s, r, u, and t denote the points $l_1 \cap k_1$, $l_n \cap k_1$, $l_1 \cap k_m$, and $l_n \cap k_m$ respectively (Figure 1). We consider the problem of finding the shortest s to t path in the arrangement formed by the lines $K \cup L$. This special case, which is mentioned as an open problem in [4], has been around for a while and has not been resolved to the best of our knowledge. We prove the following result.

Theorem 1.1 *The shortest path from s to t in the arrangement of $K \cup L$, as in Figure 1, is either the path s-r-t or the path s-u-t. A shortest s-t path never uses any edge of the arrangement that lies on one of the "middle" lines $l_2, \ldots, l_{n-1}, k_2, \ldots, k_{m-1}$. This gives us an $O(n+m)$ algorithm for computing the shortest s-t path in these classes of arrangements.*

General Arrangements: The above theorem has some consequences for the computation of the shortest path between vertices s and t of an arrangement of an arbitrary set L of n lines. Let us call a line $\ell \in L$ a *cross* line if ℓ does not pass through s or t but intersects the segment $\overline{st}$; otherwise we call ℓ an *exterior* line. Let $\mathcal{P}$ be the cell in the arrangement of the exterior lines that contains $\overline{st}$. (We are assuming the non-trivial case where no line in L contains both s and t.) If we have three cross lines l_1, l_2 and l_3 such that the intersection points $l_1 \cap l_2$, $l_2 \cap l_3$, and $l_1 \cap l_3$ are all contained in $\mathcal{P}$, we can throw away one of the three lines without affecting the length of the shortest path. This observation follows by applying Theorem 1.1 but not via a

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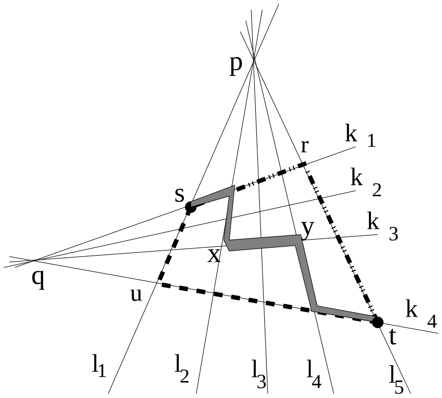

Figure 1: *The shortest s-t path is always one of s-u-t or s-r-t.*

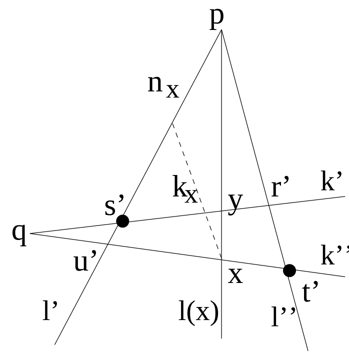

Figure 2: *The line $l(x)$ is any arbitrary line in the open cone defined by the lines l_1 and l_2.*

short argument, so we omit the argument here. By repeated application of this observation, we can compute in $O(n \log n)$ time a subset $L' \subseteq L$ such that the shortest path in the arrangement of L' has the same length as the original shortest path and the cross lines in L' can be partitioned into two sets such that no two lines from the same set intersect within $\mathcal{P}$. Note that these simplifications complement the ones in [3]. While they yield improved algorithms in very special cases, it is not clear how they help in the general case.

2 Proof of Theorem 1.1

Let l' and l'' be two lines from the pencil L through p and let k' and k'' be two lines from the pencil K through q, such that $\alpha(l') < \alpha(l'')$ and $\alpha(k') < \alpha(k'')$. Let s', t', r', and u', denote the points $l' \cap k'$, $l'' \cap k''$, $l'' \cap k'$, and $l' \cap k''$, respectively. For any point x on the segment $\overline{u't'}$, let $l(x)$ denote the line through p and x, and y denote the intersection point of k' and $l(x)$ (Figure 2). We would like to find the shortest path from s' to t' using only the lines l', l'', k', k'' and $l(x)$. We have the following lemma.

Lemma 2.1 *The shortest s'-t' path in the arrangement defined by l', l'', k', k'' and $l(x)$ does not use the edge yx on line $l(x)$. The shortest s'-t' path is one of s'-u'-t' or s'-r'-t'.*

Proof: We first prove the above lemma in those arrangements where the length of the path s'-u'-t' = length of the path s'-r'-t'. Let us call this the special case. Then we show that the special case implies the general case. The point s' splits the line k' into two rays; let ρ' denote the ray that does not conain q. Let k_x be the point on ρ' such that $|\overline{xk_x}| + |\overline{k_x s'}| = |\overline{s'u'}| + |\overline{u'x}|$. (It is easily checked that there is exactly one point on ρ' with this property.) Define n_x to be the point of intersection of l' with the line passing through x and k_x (Figure 2).

Claim: The point k_x lies on the open segment $\overline{s'y}$, or equivalently, the point n_x lies on the open segment $\overline{s'p}$.

The proof of this crucial claim is somewhat technical and is given in the Appendix. Let us now see how the special case follows from the claim.

The paths $P_1 = s'$-u'-t' and $P_2 = s'$-r'-t' (Figure 2) have equal length. Suppose $P_3 = s'$-y-x-t' has length shorter than or equal to P_1 or P_2. We know from the Claim that k_x lies on the open segment $\overline{s'y}$. Since P_1 and P_3 share the segment $\overline{xt'}$, we have $|\overline{s'y}| + |\overline{yx}| \leq |\overline{s'u'}| + |\overline{u'x}|$. We have by defintion that $|\overline{xk_x}| + |\overline{k_x s'}| = |\overline{s'u'}| + |\overline{u'x'}|$. From these two relations, we get $|\overline{s'k_x}| + |\overline{xk_x}| \geq |\overline{s'y}| + |\overline{yx}| = |\overline{s'k_x}| + |\overline{k_x y}| + |\overline{yx}|$, which violates the triangle inequality $|\overline{xk_x}| < |\overline{k_x y}| + |\overline{yx}|$. This completes the proof of the special case.

We will now prove that the special case implies Lemma 2.1. For any point x on the segment $\overline{u't'}$, let $g(x)$ denote the length of the path s'-$y(x)$-x-t', where $y(x)$ is the intersection point of $l(x)$ and k'. Since the length of the path is the sum of three continuous functions, $g(x)$ is continuous. The special case implies that if $g(a) = g(b)$ for any two distinct points a and b on the segment $\overline{u't'}$, then $g(x) > g(a)$ for any x on the open segment $\overline{ab}$. (We apply the special case with the lines $l(a)$ and $l(b)$ in place of l' and l''.)

Suppose Lemma 2.1 is violated, that is, there exists a point w in the open segment $\overline{u't'}$ such that $g(w) \leq \min\{g(u'), g(t')\}$. Let c be the point u' if $g(w) < g(u')$, else let c be any point on the open segment $\overline{u'w}$. From the special case, it follows that $g(c) > g(w)$. Similarly, we pick a point d on segment $\overline{wt'}$ distinct from w such that $g(d) > g(w)$. Let δ be any number from the open interval $(g(w), \min\{g(c), g(d)\})$. Since g is continuous, there is a point a on segment $\overline{cw}$ and a point b on segment $\overline{wd}$ such that $g(a) = \delta$ and $g(b) = \delta$. Since w is on segment $\overline{ab}$ but $g(w) < \delta$, this contradicts the implication of the special case. □

It is now easy to see that Theorem 1.1 follows from Lemma 2.1. Suppose the shortest s-t path (call it $SP(s,t)$) travels along line k_1, then bends into l_i for some $2 \leq i < n$ and then bends into k_j for some $j > 1$. This contradicts Lemma 2.1 applied to the lines l_1, l_{i+1}, k_1, k_j, and l_i. Similarly, if $SP(s,t)$ first travelled down on l_1 and then turned using one of the k_i's for $2 \leq i < m$, we have a violation of Lemma 2.1. So $SP(s,t)$ does not turn at any point other than u or r, proving Theorem 1.1.

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3 Appendix

Here we prove the claim embedded within Lemma 2.1. Consider Figure 2. By performing if necessary a translation, rotation, reflection, and a scaling operation (of the form $x \to \alpha x$ for some $\alpha > 0$) we may assume that the point q is the origin, that the lines k' and k'' have the equations $y = m_1 x$, and $y = m_2 x$ respectively, where $m_1 > m_2$, and that the line l' has the equation $x = 1$. It is easy to check that the claim holds in the original setting if and only if it holds in the modified one. Now the coordinates of s' are $(1, m_1)$, u' are $(1, m_2)$, x are $(c, m_2 c)$, k_x are $(t, m_1 t)$, where $c \geq 1$ is any arbitrary value and $t \geq 1$ is a function of c. Hence, the point n_x is also a function of c, say $f(c)$ i.e., the coordinates of n_x are $(1, f(c))$. We claim that f is a monotonically increasing function of c, i.e. $\frac{df}{dc} > 0$.

We have defined k_x such that
$|\overline{xk_x}| + |\overline{k_x s'}| = |\overline{s'u'}| + |\overline{u'x}|$. This means that
$(m_1 - m_2) + (c-1)\sqrt{1+m_2^2} = (t-1)\ \sqrt{1+m_1^2} + \sqrt{(t-c)^2 + (m_1 t - m_2 c)^2}$

This yields

$$t = \frac{ac+b}{pc+q} \tag{1}$$

where

$$\begin{aligned} a &= 2[\sqrt{1+m_1^2}\sqrt{1+m_2^2} + (m_1-m_2)\sqrt{1+m_2^2} \\ &\quad -(1+m_2^2)] \\ b &= [(m_1-m_2) + \sqrt{1+m_1^2} - \sqrt{1+m_2^2}]^2 \\ p &= 2[\sqrt{1+m_1^2}\sqrt{1+m_2^2} - (1+m_1 m_2)] \\ q &= 2[(1+m_1)^2 + (m_1-m_2)\sqrt{1+m_1^2} \\ &\quad -\sqrt{1+m_1^2}\sqrt{1+m_2^2}] \end{aligned}$$

Note that when $c = 1$, $t = 1$. Thus $a + b = p + q$. The equation of the line through x and k_x is

$$y - m_2 c = \frac{(m_1 t - m_2 c)(x - c)}{t - c}$$

We need the height (y-coordinate) of n_x, which is obtained by substituting the value $x = 1$ in the above to get

$$y = f(c) = m_2 c + \frac{(m_1 t - m_2 c)(1-c)}{t-c}.$$

Using (1), we get

$$t - c = \frac{(ac+b) - c(pc+q)}{pc+q}$$

and the numerator of the right hand side of this equality can be simplified as $-pc^2 + (a-q)c + b = -pc^2 + (p-b)c + b = (b+pc)(1-c)$.

Substituting the above in the equation for $f(c)$ and using (1), we get

$$f(c) = \frac{c(m_2 b - m_2 q + m_1 a) + b m_1}{pc + b}$$

Differentiating the above function, we get the numerator of $\frac{df}{dc}$ as
$(m_2 b - m_2 q + m_1 a)(pc + b) - p(c(m_2 b - m_2 q + m_1 a) + b m_1) = pc(m_2 b - m_2 q + m_1 a) + b(m_2 b - m_2 q + m_1 a) - pc(m_2 b - m_2 q + m_1 a) - pbm_1 = b(m_2 b - m_2 q + m_1 a - m_1 p)$

Since $a + b = p + q$, $b(m_2 b - m_2 q + m_1 a - m_1 p) = b(-m_2(a-p) + m_1(a-p))$.

We therefore have

$$\frac{df}{dc} = \frac{b(m_1 - m_2)(a - p)}{(pc + b)^2}$$

Since $b = [(m_1 - m_2) + \sqrt{1 + m_1^2} - \sqrt{1 + m_2^2}]^2$, and $(a - p) = (m_1 - m_2)(\sqrt{1 + m_2^2} - m_2)$, $\frac{df}{dc} > 0$. Hence f is an increasing function of c.

The point x has coordinates $(c, m_2 c)$ and say the coordinates of t' are $(d, m_2 d)$, where $c < d$. The coordinates of n_x are $(1, f(c))$ and p are $(1, f(d))$ (since p is the same as $n_{t'}$). Since f is an increasing function, we have $f(c) < f(d)$, which implies that the point n_x lies on the open segment $\overline{s'p}$. $\square$