Estimation of the necessary number of points in Riemannian Voronoi diagram

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Abstract

G. Leibon and D. Letscher showed that for general and *sufficiently dense point set* its Delaunay triangulation and Voronoi diagram in Riemannian manifold exist. They also proposed an algorithm to construct them for a given set.

In this paper we estimate the necessary number of points for computing the Voronoi diagram in the manifold by using sectional curvature of the manifold. Moreover, we show how many Voronoi regions exist around a Voronoi region.

1 Introduction

A Riemannian manifold (M, g) is a manifold M with its Riemannian metric g. In the manifold the *canonical distance* is induced by metric g. *Voronoi diagram* in Riemannian manifold M is defined by using the distance, i.e., for a given set of points (each point is called *site*) the manifold M is divided into regions such that each region, called *Voronoi region*, contains a site, and any points in the region is the nearest to the site rather than to any other sites of the set. The Voronoi diagram in Euclidean space has been investigated well and related results are collected in [3]. Voronoi diagrams in some Riemannian manifolds are also studied, for example, orbifold [2], Hadamard manifold [4]. G. Leibon and D. Letscher showed that the Voronoi diagram for *sufficiently dense set* in *general* Riemannian manifold exists [1]. In addition, the duality between the Voronoi diagram and the Delaunay triangulation was shown.

Their result indicates that if *many* points are given in the manifold, then Voronoi diagram and Delaunay triangulation can be computed. In [1] sufficiently dense point set is defined and characterized some lemmas. In this paper we investigate such set and consider the following problems: How to compute sufficiently dense set for a given manifold? How many points of the set are needed for a given manifold? When Voronoi diagram for the set is constructed, how many Voronoi regions are adjacent to a region?

From the point of view of application, above problems are regarded as the problem of the surface construction. Consider the given surface as the Riemannian manifold. Our problems are rewritten: How to compute a set of points such that the set is good approximation using Delaunay triangulation for the given surface? How many points of the set are needed for such the approximation? How many points are adjacent to a point in the approximation? So, our problems are meaningful in the spaces with curvatures.

In this paper, we define an *ε-packing-covering (ε-PC) set* (Section 3.1), a similar concept is considered in [5]. The ε-PC set is sufficiently dense and is easily computed for a given positive constant ε. We show that the number of points is bounded by upper and lower curvatures of a given manifold and given positive constant ε (Section 3.2). Suppose the Voronoi diagram for the ε-PC set, the number of adjacent Voronoi regions of a Voronoi region is estimated (Section 3.3).

2 Preliminaries

2.1 From computational geometry

In this subsection we describe the results in [1]. Their results are based on a *generic* and *sufficiently dense* set of points in a given Riemannian manifold. These concepts are defined as follows:

Definition
Let M be a d-dimensional manifold. A set of points in M is said to be *generic* if any $d+2$ points do not lie on the sphere in M.

Let $\mathcal{P} \subset M$ be a finite set of points. For any point $y \in M$ and $z \in B_{4\mathrm{rad}(y)}(y)$, if $B_{\mathrm{rad}(y)}(z)$ contains a point of $\mathcal{P}$, $\mathcal{P}$ is said to be *sufficiently dense*, where $B_r(x)$ is the ball with center x in the M and $\mathrm{rad}(x) := \frac{1}{5}\times$ (strong convexity radius[1] of M at x).

The former is well-known, the latter is a new concept and characterized by next lemma in [1].

Lemma A Let M be a given Riemannian manifold and $\mathcal{P}$ be a set of points on M. Let K be a positive upper bound on the sectional curvature of manifold M and $\mathrm{inj}(M)$ be an injective radius[2] of M. Let $r = \min\left\{\frac{\mathrm{inj}(M)}{10}, \frac{\pi}{10\sqrt{K}}\right\}$. If every ball of radius r in M

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[1] A positive number r is said to be *strong convexity radius* of M at x if for any two points in $B_r(x)$ a unique shortest geodesic segment connecting the points exists and the geodesic is contained in $B_r(x)$.

[2] The *injective radius* r of M at x is the largest radius such that $B_r(x)$ can be embedded as a ball. The *injective radius* of M is the infimum of the injective radii at each point.

contains a point in $\mathcal{P}$, then the points are sufficiently dense.

The sufficiently small constant r depending upon injective radius and sectional curvature of M is fixed. For any point in the manifold M there exists a point in $\mathcal{P}$ such that the distance between the points is less than r. If the number of the set $\mathcal{P}$ increases sufficiently, above situation in Lemma A can be achieved. However, it is not good that the number is increasing endlessly when the set is constructed. So, the upper and lower bound is considered in this paper.

G. Leibon and D. Letscher showed the following theorem:

Lemma B Let M be a given Riemannian manifold and $\mathcal{P} := \{p_1, \cdots, p_n\}$ be a set of points on M. If $\mathcal{P}$ is a generic and sufficiently dense set of points, then there exists a unique Delaunay triangulation with vertices $\{p_1, \cdots, p_n\}$.

From this lemma, if a generic and sufficiently dense set is constructed, then its Delaunay triangulation exists. In addition, they proved the duality between Delaunay triangulation and Voronoi diagram. Thus they showed the existence of Voronoi diagram for the set in the given Riemannian manifold.

2.2 From Riemannian geometry

In this subsection some results in the Riemannian geometry are described. We describe two results: Bishop's theorem and Myers' theorem.

Bishop's theorem ([5], p155) is a kind of comparison theorem, which is well-known in the Riemannian geometry. The volume of ball in different manifolds is compared by Bishop's theorem. The statement is as follows:

Lemma C (Bishop's theorem) Let M be a d-dimensional complete[3] Riemannian manifold and M_δ^d be a d-dimensional complete simply connected Riemannian manifold with constant sectional curvature δ. Let $k(p)$ be a sectional curvature at p of M and $B_r(p)$ be a r-ball with center p in M.

If $k(p) \geq \delta$ for any point p in M and $0 < r \leq \pi/\sqrt{\delta}$, then

$$\mathrm{vol}B_r(p) \leq v_r(\delta),$$

where $\mathrm{vol}B_r(p)$ is the volume of $B_r(p)$ in M and $v_r(\delta)$ is the volume of r-ball in M_δ^d. Similarly, opposite inequality of volumes is settled when the inequality of curvatures is opposite.

In above lemma, if the curvature δ is negative, then it is possible that r becomes infinity.

Using Bishop's theorem, the volume of r-balls in a given manifold can be compared to the volume of r-ball in another manifold if there exist the inequality of sectional curvatures. In above Lemma, another manifold is replaced with the manifold with constant curvature.

[3] For any point $p \in M$ and any $r > 0$, the r-ball $B_r(p)$ is compact.

The diameter of manifold M, $d(M) := \sup\ \{d(p,q); p, q \in M\}$, is bounded by the lowest curvature κ of M.

Lemma D (Myers' theorem, [5] p.102) Let M be a complete Riemannian manifold and p be a point in M. Let $k(p)$ be a sectional curvature at p in M and κ be a positive constant. If $k(p) \geq \kappa > 0$ for every point p, then M is compact and $d(M) \leq \pi/\sqrt{\kappa}$.

3 Estimation of the number of points

3.1 ε-packing-covering set

Firstly, the *packing-covering set* is defined.

Definition 1 *A finite set $\mathcal{P} := \{p_1, \cdots, p_n\}$ of points in a compact[4] Riemannian manifold M is said to be ε-*packing-covering *for a positive constant ε, denoted by ε-PC, if intersection of any two ε-balls with center p_i and with p_j is empty and the union of all 2ε-balls with center p_i is covering of M.*

The condition of ε-PC set is the following:

1. $B_\varepsilon(p_i) \cap B_\varepsilon(p_j) = \emptyset$ for any i, j.
2. $\cup_{i=1}^n B_{2\varepsilon}(p_i) = M$.

Similar set is familiar to in Riemannian geometry.

We show that ε-PC set always exists for given ε.

Lemma 1 *For any positive constant ε and for any compact Riemannian manifold, there exists an ε-PC set.*

Proof: Let M be a compact Riemannian manifold. It is trivial that there is a finite set of points such that all neighborhoods is covering of M because of compactness of M.

So, we show the finite set $\mathcal{P}$ satisfies the conditions above. Suppose there exist a point p in M and does not included in the union of 2ε neighborhoods of points in $\mathcal{P}$. The distance between p and any point in $\mathcal{P}$ is greater than 2ε. Since the ε-ball with center p does not intersect with any other ε-balls, we can add p to $\mathcal{P}$. It is possible to repeat above step if such a point exists in M. Otherwise, the set is ε-PC. $\square$

An construction algorithm is obtained from above lemma.

When this algorithm is performed, suppose two oracles such that 1) a point is selected from a given manifold M; 2) for any point p in M and positive number r, the ball $B_r(p)$ is computed. Since the manifold is compact, $\mathcal{P}$ becomes finite. So, the complexity of this algorithm is linear for the number of points $\mathcal{P}$.

3.2 For a manifold

In this subsection we consider estimation of the number of necessary points of Voronoi diagram in a given Riemannian manifold.

Let M be a complete compact Riemannian manifold. Consider an ε-PC set on M where ε is a positive constant less than or equal to $\frac{r}{2}$.

[4] A manifold is compact if the manifold are covered by neighborhoods of *finite* set.

Algorithm 1 Construction algorithm for ε-PC set on a given compact Riemannian manifold M

Input: efficiently small ε, given manifold M
Output: ε-PC set $\mathcal{P}$

1. Initialize: $\mathcal{P} := \emptyset$, $C := \emptyset$;
2. while (C does not cover M) **do** {
  choose p from $M \setminus C$;
  $\mathcal{P} := \mathcal{P} \cup \{p\}$;
  $C := C \cup B_{2\varepsilon}(p)$;
}
3. return $\mathcal{P}$;

Lemma 2 *Let r be the positive constant number for manifold in Lemma A. If $\varepsilon \leq \frac{r}{2}$, called ε is* sufficiently small, *then any ε-PC set is sufficiently dense.*

Proof: We show that every ball of radius r in M contains a point of an ε-PC set. Then we show that the ε-PC set is sufficiently dense.

Above statement is shown. Because the set is ε-PC, the manifold M is covered by 2ε-balls. In other words, for every point in M there exists a point in the ε-PC set such that the distance between two points is less than 2ε. Since $r \geq 2\varepsilon$, any ball of radius r contains one or more points in ε-PC set. □

Consider an ε-PC set $\mathcal{P}$ on M. From Lemma B, there exist a Delaunay triangulation and a Voronoi diagram for $\mathcal{P}$.

Theorem 1 *If a set of points $\mathcal{P}$ on a given Riemannian manifold is a generic ε-PC set for sufficiently small ε, then there exists a unique Delaunay triangulation (Voronoi diagram) for $\mathcal{P}$.*

So, we get an ε-PC set and its Delaunay triangulation (Voronoi diagram) exists for sufficiently small positive constant ε. Since ε gives a measure of mesh, the more ε decrease, the more the number of points increases. We show the relation between ε and the number of points in the ε-PC set by Bishop's theorem.

Theorem 2 *Let $\mathcal{P}$ be a generic ε-PC set for sufficiently small ε on a given compact manifold M. Let n be the number of points in $\mathcal{P}$. The number n satisfies inequalities below:*

$$\frac{\mathrm{vol}M}{V_{2\varepsilon}} \leq n \leq \frac{\mathrm{vol}M}{v_{\varepsilon}},$$

where $V_{2\varepsilon}$ (v_ε) is upper(lower) bound of the volume of 2ε-ball (ε-ball) among balls with center in $\mathcal{P}$, respectively.

Proof: Suppose an ε-PC set on M. From the property of ε-PC set, all ε-balls with center point in $\mathcal{P}$ do not intersect each other. All ε-balls are included in M. This inequality is settled: $\mathrm{vol}M \geq \sum_{p\in\mathcal{P}} \mathrm{vol}B_\varepsilon(p) \geq n \cdot v_\varepsilon$.

Conversely, the union of all 2ε-balls cover M. The next inequality is shown: $\mathrm{vol}M \leq \sum_{p\in\mathcal{P}} \mathrm{vol}B_{2\varepsilon}(p) \leq n \cdot V_{2\varepsilon}$. □

When the sectional curvature of M is bounded, the result above is more precisely.

Corollary 1 *Consider the situation in Theorem 2. If the manifold is complete and the sectional curvature k of M is bounded by $0 \leq \kappa \leq k \leq K$, then,*

$$n \leq v_{\pi/\sqrt{\kappa}}(\kappa)/v_\varepsilon(K),$$

where $v_r(k)$ is the volume of r-ball in a complete simply connected Riemannian manifold with constant sectional curvature k.

Proof: See Appendix A. □

In addition, this number is evaluated by computing the volume in appendix B.

Corollary 2 *Suppose the situation in Corollary 1. The number of points n is bounded:*

$$n \leq \begin{cases} \frac{d!!}{(d-1)!!} \cdot \frac{\pi}{\kappa^{d/2}\varepsilon^d} & d \text{ is odd} \\ \frac{d!!}{(d-1)!!} \cdot \frac{2}{\kappa^{d/2}\varepsilon^d} & d \text{ is even,} \end{cases}$$

where $(2p)!! = (2p)(2p-2)\cdots 2, (2p+1)!! = (2p+1)(2p-1)\cdots 1$ for positive integer p.

Proof: Compute above inequality in Corollary 1 by using equation (2), (3) and (4) in Appendix B. □

3.3 For a point

In the previous section, the number of points for Riemannian Voronoi diagram in manifold M is described. In this section we show that the number of Voronoi regions adjacent to *a Voronoi region* when consider the Voronoi diagram for general and ε-PC set of points.

Suppose an ε-PC set of points in M for sufficiently small ε. The following lemma about the distance between two adjacent points in Voronoi diagram is shown.

Lemma 3 *Consider an ε-PC set $\mathcal{P}$ for sufficiently small ε. Consider two points such that Voronoi regions of the points are adjacent in Voronoi diagram for $\mathcal{P}$. Then, the distance between the points is grater than or equal to 2ε and less than or equal to 4ε.*

Proof: See Appendix A. □

We estimate the number of adjacent point around a point by this lemma.

Theorem 3 *Consider an ε-PC set $\mathcal{P}$ for sufficiently small ε on a complete compact Riemannian manifold M. Let κ and K be the lower and upper bound of sectional curvature of M. Let m be the number of adjacent regions of a Voronoi region of a point p in the Voronoi diagram for $\mathcal{P}$. The following relation is settled.*

$$\frac{v_{2\varepsilon}(K)}{v_{2\varepsilon}(\kappa)} - 1 \leq m \leq \frac{v_{5\varepsilon}(\kappa)}{v_\varepsilon(K)} - 1,$$

where $v_r(k)$ is the volume of r-ball in the Riemannian manifold with constant curvature k.

Proof: Let $\mathcal{Q}$ be a set of point whose Voronoi region is adjacent with the region of p.

Let $B_{5\varepsilon}(p)$ be the 5ε-ball with center point p. Since the distance between the points whose Voronoi regions are adjacent is less than 4ε from above lemma, the 5ε-ball contains all ε-balls with center point q in $\mathcal{Q}$. So, $B_{5\varepsilon}(p)$ contains $\|\mathcal{Q} \cup q\| = (m+1)$ ε-balls.

$$(m+1)v_\varepsilon(K) \leq \mathrm{vol}B_{5\varepsilon}(p) \leq v_{5\varepsilon}(\kappa).$$

The last inequality is also shown by Lemma C. From these inequalities we get the following evaluation:

$$m \leq v_{5\varepsilon}(\kappa)/v_\varepsilon(K) - 1.$$

In the case of lower bound, consider the union of 2ε-ball with center q in $\mathcal{Q} \cup \{p\}$, denoted by $\tilde{M}$. Consider a ball with center p such that the r-ball is included in $\tilde{M}$. If $r = 2\varepsilon$, then this ball is always included in $\tilde{M}$. The following relation is settled:

$$v_{2\varepsilon}(K) \leq \mathrm{vol}\tilde{M} \leq (m+1) \cdot v_{2\varepsilon}(\kappa).$$

The lower bound is proved by this inequality. □

[**Remark**] Since $\kappa \leq K$ under the condition of Theorem 3, then $v_{2\varepsilon}(\kappa) \geq v_{2\varepsilon}(K)$ is settled from the volume of 2ε-ball. So, the lower bound of the above theorem is always negative number. It gives trivial bound.

In addition, if the lowest curvature κ is positive, then the number is bounded by the sufficiently small constant ε, the dimension d and the curvature κ.

Corollary 3 *Suppose the situation in Theorem 3. If the lower bound κ of curvature is positive, then the number m of adjacent region is evaluated:*

$$\frac{1-4C(d)\varepsilon^2 K}{1-4C(d)\varepsilon^2\kappa} - 1 \leq m \leq \frac{5^d - 5^{d+2}C(d)\varepsilon^2\kappa}{1-C(d)\varepsilon^2 K} - 1,$$

where $C(d) = d(d-1)/\{6(d+2)\}$.

Proof: Compute above inequality in Theorem 3 by using (4) in Appendix B. □

4 Conclusion

We show some results about ε-PC set in a complete simply connected compact Riemannian manifold. In this section these results are adapted to a manifold with constant curvature. Consider $K = \kappa > 0$ and d is odd in Corollary 2 and Corollary 3, respectively:

$$n < \frac{d!!}{(d-1)!!} \cdot \frac{\pi}{\kappa^{d/2}\varepsilon^d}, \; 0 \leq m \leq \frac{5^d - 5^{d+2}C(d)\varepsilon^2\kappa}{1-C(d)\varepsilon^2\kappa} - 1.$$

If $r = \pi/(10\sqrt{\kappa})$ in Lemma A, it is possible that $\varepsilon = \pi/(20\sqrt{\kappa})$ by Lemma 2. Suppose $d = 3$, then above inequalities can be computed:

$$n \leq 3/2 \cdot 20^3/\pi^2 \sim 1217.0879, \quad m \leq 109.12219.$$

This number is not so good, but it is possible that the evaluation is more better if $(\varepsilon, k\varepsilon)$-packing-covering set for a given manifold exists. The set is defined as follows:

Definition 2 *A finite set $\mathcal{P} := \{p_1, \cdots, p_n\}$ of points in a compact Riemannian manifold M is said to be $(\varepsilon, k\varepsilon)$-*packing-covering *for positive constants ε and $k (> 1)$, denoted by $(\varepsilon, k\varepsilon)$-PC, if intersection of any two ε-balls with center p_i and with p_j is empty and the union of all $k\varepsilon$-balls with center p_i is covering of M.*

For a given manifold, if $(\varepsilon,\ k\varepsilon)$-PC set exists, then the distance between the points whose Voronoi regions are adjacent in the $(\varepsilon,\ k\varepsilon)$-PC set is between 2ε and $2k\varepsilon$. This relation is applied to the proof of Theorem 3. All the points around a point are contained in $(2k+1)\varepsilon$-ball. So, the inequality in Corollary 3 is improved:

$$m \leq \frac{(2k+1)^d - (2k+1)^{d+2}C(d)\varepsilon^2\kappa}{1-C(d)\varepsilon^2 K} - 1.$$

In actual, consider regular triangle lattice in Euclidean plane, which is the set of vertex of regular triangle and the distance between any adjacent points is 2ε. Such the set of points is $\left(\varepsilon, \frac{2}{\sqrt{3}}\varepsilon\right)$-PC. The coefficient of above inequality becomes $m \sim 7.5032621$

Consequently, the number of adjacent Voronoi regions of a Voronoi region is less than or equal to 7. This number is better rather than 22.763757 in case of $(\varepsilon, 2\varepsilon)$-PC set. If a small positive number k is found for given manifold, this bound can be improved.

References

[1] G. Leibon and D. Letscher, "Delaunay triangulations and Voronoi diagrams for Riemannian manifolds," *Proceeding 16th Annual ACM Symposium on Computational Geometry*, 2000, pp. 341-349.

[2] M. Mazón and T. Recio, "Voronoi Diagrams on Orbifolds," *Computational Geometry : Theory and Applications*, vol. 8, No. 5, 1997, pp. 219-230.

[3] A. Okabe, B. Boots, K.Sugihara and S. N. Chiu, "Spatial Tessellations: Concepts and Applications of Voronoi Diagrams," Second Edition, John Weily, New York, 2000.

[4] K. Onishi, J. Itoh: "Voronoi Diagram in Simply Connected Complete Manifold", IEICE Transactions on Fundamentals, vol. E85-A No5, pp.944-948, 2002.

[5] T. Sakai, "Riemannian Geometry," Translation of Mathematical Monographs, vol. 149, Amer. Math. Soc., 1996.

A Proofs of Lemma and Corollary

Proof of Corollary 1 Let $d(M)$ be the diameter of the given manifold M. This diameter is bounded by Lemma D:

$$d(M) \le \frac{\pi}{\sqrt{\kappa}}.$$

Since the diameter is the largest distance between any two points in M, consider $d(M)$-ball with any point p in M. Then this ball contains M. So, the volume of M is bounded by

$$\mathrm{vol}M \le B_{d(M)}(p) \le B_{\pi/\sqrt{\kappa}}(p)$$

Apply Lemma C to the volume of the ball. Suppose $r = \pi/\sqrt{\kappa}$ and a complete simply connected Riemannian manifold M^d_κ with constant sectional curvature κ. Since for any point in M sectional curvature k is grater than or equal to κ, we can use Lemma C. The volume of the ball is less than or equal to $v_{\pi/\sqrt{\kappa}}(\kappa)$. Then, we get the following inequality:

$$\mathrm{vol}M \le v_{\pi/\sqrt{\kappa}}(\kappa).$$

The volume of v_ε is also evaluated by Lemma C. In this case, suppose a complete simply connected Riemannian manifold M^d_K with constant sectional curvature K. Since K is the upper bound of the sectional curvature, for any point of M the sectional curvature k of the point is less than or equal to K. Moreover the ε is smaller than $\pi/\sqrt{\kappa}$. So, we get the following inequality $v_\varepsilon \ge v_\varepsilon(K)$.

Finally, the number of points is bounded by

$$n \le \frac{\mathrm{vol}M}{v_\varepsilon} \le \frac{v_{\pi/\sqrt{\kappa}}(\kappa)}{v_\varepsilon(K)}.$$

□

Proof of Lemma 3 Let p, p' be two adjacent points in the lemma above. Since the Voronoi regions of p and of p' are adjacent, there exists an equidistant point from p and from p' such that the point is shared point between the Voronoi regions of p and of p'.

Suppose the distance $d(p,p') > 4\varepsilon$. Let q be the most nearest point p and p' among equidistant points from p and p' (this point is unique because the set of points is ε-PC). This point satisfies inequalities below:

$$d(p,q) + d(p',q) = 2d(p,q) \ge d(p,p') > 4\varepsilon.$$

Thus the q is not included in $B_{2\varepsilon}(p)$ nor $B_{2\varepsilon}(p')$. From the condition of ε-PC set, there exists a point $p''(\neq p,p')$ in $\mathcal{P}$ such that $B_{2\varepsilon}(p'')$ contains q. So, q is not shared point between Voronoi regions of p and of p'.

Consider another point q' which is equidistant point from p and from p'. Because the q is most nearest point from p, p',

$$d(p,q) < d(p,q') \text{ and } d(p',q) < d(p',q').$$

So, the distance between q' and $p(p')$ is larger than 2ε. The q' is not share point also. Thus, no shared point between the Voronoi regions of p and of p' exist. This is contradiction with that the Voronoi regions of p and of p' are adjacent.

Suppose the distance between p, p' is less than 2ε. $B_\varepsilon(p)$ and $B_\varepsilon(p')$ has intersection. This is contradiction with the property of ε-PC set. □

B Estimation of Volume of r-ball with constant curvature

The volume of r-ball $v_r(\delta)$ in M^d_δ is expressed from some equations in [5]:

$$v_r(\delta) \;=\; \int_{S^{d-1}} \mathrm{d}S^{d-1} \int_0^r s_\delta^{d-1}(t)\mathrm{d}t,$$

where

$$s_\delta^{d-1}(t) = \begin{cases} \left\{(\sin\sqrt{\delta}t)/\sqrt{\delta}\right\}^{d-1} & \delta > 0 \\ t^{d-1} & \delta = 0 \\ \left\{(\sinh\sqrt{|\delta|}t)/\sqrt{|\delta|}\right\}^{d-1} & \delta < 0 \end{cases}.$$

The case of $\delta > 0$ is computed in this paper.

$$\begin{aligned} v_r(\delta) &= \int_{S^{d-1}} \mathrm{d}S^{d-1} \int_0^r \left\{\sin\sqrt{\delta}t/\sqrt{\delta}\right\}^{d-1} \mathrm{d}t \\ &= \int_{S^{d-1}} \mathrm{d}S^{d-1} \int_0^{\sqrt{\delta}r} \sin^{d-1} x\mathrm{d}x \cdot \delta^{-d/2}. \end{aligned}$$

Using this formulation, the above volume is bounded.

$$\begin{aligned} \int \sin^{2p} x\mathrm{d}x = &-\cos x\left[\frac{\sin^{2p-1}x}{2p} + \frac{(2p-1)\sin^{2p-3}x}{2p(2p-2)} \right. \\ &\left. + \cdots + \frac{(2p-1)!!}{(2p)!!}\sin x\right] + \frac{(2p-1)!!}{(2p)!!}x, \end{aligned}$$

$$\begin{aligned} \int \sin^{2p+1} x\mathrm{d}x = &-\cos x\left[\frac{\sin^{2p}x}{2p+1} + \frac{2p\sin^{2p-2}x}{(2p+1)(2p-1)} \right. \\ &\left. + \cdots + \frac{(2p)!!}{(2p+1)!!2}\sin^2 x + \frac{(2p)!!}{(2p+1)!!}\right], \end{aligned}$$

where $(2p)!! = (2p)\cdot(2p-2)\cdots 2$, $(2p+1)!! = (2p+1)\cdot(2p-1)\cdots 1$.

Firstly, $d = 2p+1$ case is considered.

$$\begin{aligned} &\int_0^{\sqrt{\delta}r} \sin^{d-1} x\mathrm{d}x \\ &= \left[-\cos x\left\{\frac{\sin^{2p-1}x}{2p} + \frac{(2p-1)\sin^{2p-3}x}{2p(2p-2)} \right.\right. \\ &\qquad \left.\left. + \cdots + \frac{(2p-1)!!}{(2p)!!}\sin x\right\} + \frac{(2p-1)!!}{(2p)!!}x\right]_0^{\sqrt{\delta}r} \\ &= -\cos\sqrt{\delta}r\left\{\frac{\sin^{2p-1}\sqrt{\delta}r}{2p} + \frac{(2p-1)\sin^{2p-3}\sqrt{\delta}r}{2p(2p-2)} \right. \\ &\qquad \left. + \cdots + \frac{(2p-1)!!}{(2p)!!}\sin\sqrt{\delta}r\right\} + \frac{(2p-1)!!}{(2p)!!}\sqrt{\delta}r. \quad (1) \end{aligned}$$

The first term of (1) is non-positive when $0 < r \le \frac{\pi}{\sqrt{\delta}}$. So, the volume is bounded by the second term of (1). The following bound is shown:

$$\int_0^{\sqrt{\delta}r} \sin^{d-1} x \mathrm{d}x \le \frac{(d-2)!!}{(d-1)!!}\sqrt{\delta}r.$$

Secondly, $d = 2p + 2$ is considered.

$$\begin{aligned}\int_0^{\sqrt{\delta}r} \sin^{d-1} \mathrm{d}x = &\left[-\cos x \left\{ \frac{\sin^{2p} x}{2p+1} + \frac{2p\sin^{2p-2} x}{(2p+1)(2p-1)} \right.\right. \\ &\left.\left. + \cdots + \frac{(2p)!!}{(2p+1)!!\cdot 2}\sin^2 x + \frac{(2p)!!}{(2p+1)!!}\right\}\right]_0^{\sqrt{\delta}r} \\ = &-\cos\sqrt{\delta}r \left\{ \frac{\sin^{2p}\sqrt{\delta}r}{2p+1} + \frac{2p\sin^{2p-2}\sqrt{\delta}r}{(2p+1)(2p-1)} \right. \\ &\left. + \cdots + \frac{(2p)!!}{(2p+1)!!\cdot 2}\sin^2\sqrt{\delta}r + \frac{(2p)!!}{(2p+1)!!}\right\} \\ &+ \frac{(2p)!!}{(2p+1)!!}.\end{aligned}$$

The upper bound of the volume is shown:

$$\int_0^{\sqrt{\delta}r} \sin^{d-1} \mathrm{d}x < \frac{(d-2)!!}{(d-1)!!}\left(1 - \cos\sqrt{\delta}r\right).$$

So, the volume is bounded by

$$v_r(\delta) < \int_{S^{d-1}} \mathrm{d}S^{d-1} \cdot \frac{(d-2)!!}{(d-1)!!} \cdot \sqrt{\delta}r \cdot \delta^{-d/2} \qquad d \text{ is odd,} \tag{2}$$

$$v_r(\delta) < \int_{S^{d-1}} \mathrm{d}S^{d-1} \cdot \frac{(d-2)!!}{(d-1)!!} \times \left(1 - \cos\sqrt{\delta}r\right) \cdot \delta^{-d/2} \qquad d \text{ is even.} \tag{3}$$

For sufficiently small radius r, the evaluation of the volume of r-ball is possible.

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + O(x^5), \\ \sin^d x &= \left(x - \frac{x^3}{6} + O(x^5)\right)^d \\ &= x^d - \frac{d}{6}x^{d+2} + O(x^{d+4}).\end{aligned}$$

Then above equation is integrated, the following is settled.

$$\begin{aligned}\int_0^{\sqrt{\delta}r} \sin^d x \mathrm{d}x = &\left[\frac{x^{d+1}}{d+1} - \frac{dx^{d+3}}{6(d+3)} + O(x^{d+5})\right]_0^{\sqrt{\delta}r} \\ = &\frac{r^{d+1}}{d+1}\delta^{(d+1)/2} - \frac{dr^{d+3}}{6(d+3)}\delta^{(d+3)/2} \\ &+ O(r^{d+5}\delta^{(d+5)/2}).\end{aligned}$$

$$\begin{aligned}v_r(\delta) \sim &\int_{S^{d-1}} \mathrm{d}S^{d-1} \cdot \delta^{-d/2} \\ &\times \left(\frac{r^d}{d}\delta^{d/2} - \frac{(d-1)r^{d+2}}{6(d+2)}\delta^{(d+2)/2} + O(r^{d+4}\delta^{(d+4)/2})\right) \\ \sim &\int_{S^{d-1}} \mathrm{d}S^{d-1} \left(\frac{r^d}{d} - \frac{(d-1)r^{d+2}}{6(d+2)}\delta + O(r^{d+4}\delta^2)\right).\end{aligned} \tag{4}$$