Nonorthogonal Polyhedra Built from Rectangles

Abstract[*]

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We prove that any polyhedron of genus zero or genus one built out of rectangular faces must be an orthogonal polyhedron, but that there are nonorthogonal polyhedra of genus seven all of whose faces are rectangles. This leads to a resolution of a question posed by Biedl, Lubiw, and Sun [BLS99].

1 Introduction

A paper by Biedl, Lubiw, and Sun [BLS99] raised the following intriguing question: "Can an orthogonal net ever fold to a nonorthogonal polyhedron?" We answer this question here: YES for sufficiently high genus, and NO for sufficiently small genus. First we clarify the meaning of the terms in the question.

An *orthogonal net* is an orthogonal polygon: a simple, planar polygon all of whose sides meet at right angles. See, e.g., Fig. 1. To *fold* an orthogonal net

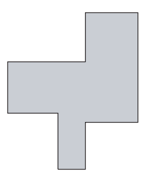

Figure 1: A simple orthogonal polygon.

means to crease it along segments parallel to the polygon's sides and close up its perimeter. Let us call this an *orthogonal folding*, to contrast it with an arbitrary folding (when the creases can be at arbitrary angles). A *polyhedron* is a surface formed of planar faces, each a simple polygon, such that any pair is either disjoint, share one vertex, or share an edge; such that each polyhedron edge is shared by exactly two faces; and such that the "link" of each vertex is a cycle. An *orthogonal polyhedron* is a polyhedron all of whose faces meet at edges with dihedral angles that are a multiple of $\pi/2$. It is known that arbitrary foldings of orthogonal polygons can lead to nonorthogonal polyhedra [LO96][DDL$^+$99], but the question obtained by restricting to orthogonal foldings seems to be new.

We display their question, phrased in a positive sense, for later reference:

> **Question 1**. If a polyhedron is created by an orthogonal folding of an orthogonal polygon, must it be an orthogonal polyhedron?

We first study a related question, which is divorced from the concept of folding:

> **Question 2**. If a polyhedron's faces are all rectangles, must it be an orthogonal polyhedron?

We should note that the definition of a polyhedron permits coplanar rectangles, so this second question could be phrased as: "if a polyhedron's faces can be partitioned into rectangles," or, "if a polyhedron can be constructed by gluing together rectangles." Both Questions 1 and 2 can be viewed as seeking to learn whether, loosely speaking, orthogonality in $\mathbb{R}^2$ forces orthogonality in $\mathbb{R}^3$.

An orthogonal folding of an orthogonal polygon produces faces that can be partitioned into rectangles. Thus if rectangle faces force orthogonal dihedral angles, then orthogonal foldings force orthogonal dihedral angles. So we have the following implications:

- Q2: YES $\Rightarrow$ Q1: YES.
- Q1: NO $\Rightarrow$ Q2: NO.

(The second is merely the contrapositive of the first.) However, it is possible that the answer to Q2 is NO but the answer to Q1 is YES, for it could be that the nonorthogonal polyhedra made from rectangles cannot be unfolded without overlap to an orthogonal net. We first explore Q2 to remove ourselves from the little-understood area of nonoverlapping unfoldings. We will see that, nevertheless, we can ultimately prove the answer to both questions is NO for polyhedra of genus

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[*]Full paper available at `http://arXiv.org/abs/cs/0110059/`. At some risk of confusion, section, figure, lemma, and theorem numberings of the full paper are retained in this condensed version.

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seven or above (Thms. 1 and 2), but YES for genus zero and one (Thm. 15).

Upon release of our report [DO01],[†] we learned that many of our discoveries had been made independently by a group at the Univ. of Waterloo.[1] Since then that group improved their results in several directions, in particular narrowing the gap in knowledge between genus 1 and 7 to 2 to 6: they establish that the answers to Q1 and Q2 are YES for genus 2 and NO for genus 6.

2 A Nonorthogonal Polyhedron made with Rectangles

Although this reverses the order of our actual development, and might appear unmotivated, we first present the example that shows that the answer to Question 2 is NO. The polyhedron is shown in two views in Fig. 2. It

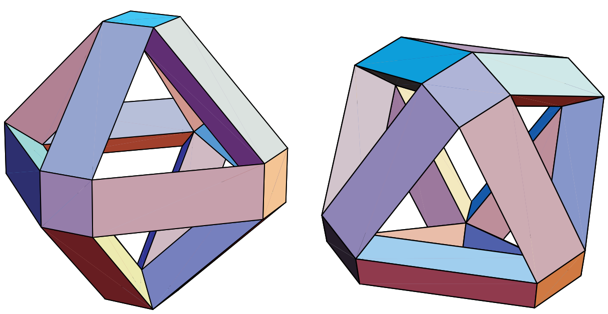

Figure 2: An nonorthogonal polyhedron composed of rectangular faces.

has the structure (and symmetry) of a regular octahedron, with each octahedron vertex replaced by a cluster of five vertices, and each octahedron edge replaced by a triangular prism. Let us fix the squares at each octahedron vertex to be unit squares. The length L of the prisms is not significant; in the figure, $L = 3$, and any length large enough to keep the interior open would suffice as well. The other side lengths of the prism are, however, crucially important. The open triangle hole at the end of each prism has side lengths 1 (to mesh with the unit square), $\sqrt{3}/2$ and $\sqrt{3}/2$; see Fig. 3.

This places the fifth vertex in the cluster displaced by 1/2 perpendicularly from the center of each square, forming a pyramid, as shown in Fig. 4.

The central isosceles right triangle (shaded in Fig. 4) guarantees that two oppositely oriented incident triangular prisms meet at right angles, just as do the corresponding edges of an octahedron. The result is a closed polyhedron of $V = 6 \cdot 5 = 30$ vertices, $E = 84$ edges, and $F = 42$ faces. The 42 faces include 6 unit squares, 12 $L \times 1$ rectangles, and 24 rectangles of size

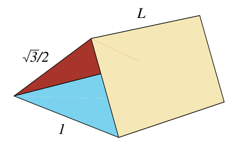

Figure 3: Right triangular prism, used twelve times in Fig. 2.

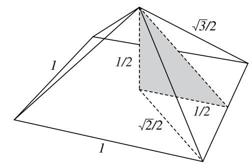

Figure 4: The cluster of five vertices replacing each octahedron vertex form a pyramid.

$L \times \sqrt{3}/2$. Clearly the polyhedron is nonorthogonal: the three long edges of each prism have dihedral angles of approximately 54.7° ($= \tan^{-1}\sqrt{2}$), 54.7°, and 70.5° ($= \pi - 2\tan^{-1}\sqrt{2}$), two adjacent prisms meet at the same 70.5° angle, and the prisms meet the squares at 45°. So not a single dihedral angle is a multiple of $\pi/2$.

By Euler's formula,

$$V - E + F = 2(1 - g) \tag{1}$$

where g is the genus. From this we can compute the genus of the surface to be $g = 7$. As it is easy to augment the polyhedron with attached (orthogonal) structures that increase the genus, this example establishes that the answer to Question 2 is NO:

Theorem 1 *There exist nonorthogonal polyhedra of genus $g \geq 7$ whose faces are all rectangles.*

Theorem 1 does not settle Question 1 immediately, for it could be that the polyhedron cannot be unfolded to an "orthogonal net." In fact, we were not successful in finding a nonoverlapping unfolding of its surface, only one that permits boundary sections to touch but not cross. Such an unfolding could be cut out of a single sheet of paper, but the cuts would have to be infinitely thin for perfect fidelity.

However, the goal usually is to obtain a simple orthogonal polygon, and such touchings violate the usual definition of a polygon. A simple modification of the example illustrated in Fig. 6 does lead to nonoverlap. Each square is replaced by the five faces of a unit cube. The polyhedron remains nonorthogonal (although now it has some $\pi/2$ dihedral angles), and it may be unfolded to the net shown in Fig. 7. The role of the cubes is to spread out the unfoldings of each prism so that they avoid overlap.

This establishes that the answer to Question 1 is also NO:

Theorem 2 *There is a simple orthogonal polygon that folds orthogonally to a nonorthogonal polyhedron.*

3 Overview of Proof

The proof that the answer to Questions 1 and 2 is YES for sufficiently small genus, largely deleted from this abstract, proceeds in three main steps:

[1] Including Therese Biedl, Timothy Chan, Erik Demaine, Martin Demaine, Graeme Kemkes, Anna Lubiw, Paul Nijjar, Ryuhei Uehara, and Ming-wei Wang.

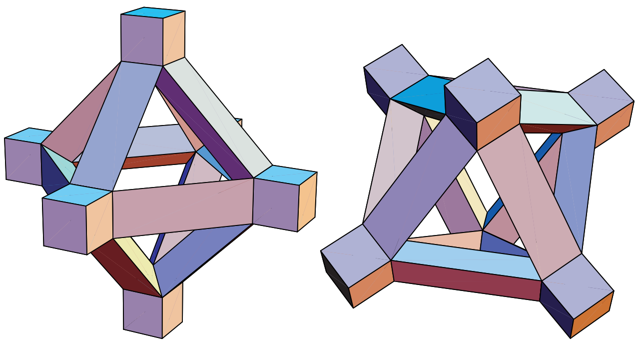

Figure 6: Adding cubes to Fig. 2.

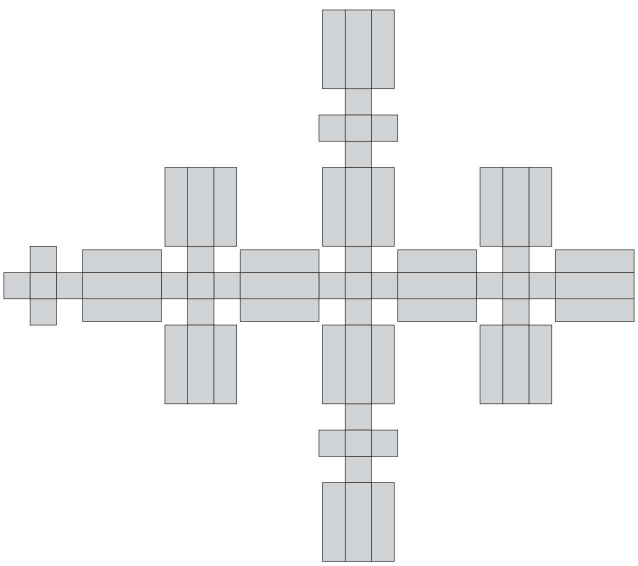

Figure 7: An orthogonal polygon that folds to the nonorthogonal polyhedron in Fig. 6.

1. Sec. 4 establishes constraints on low-degree vertices, showing how orthogonal and nonorthogonal dihedral angles may mix locally at one vertex.
2. Sec. 5 (omitted in this version) derives a corollary to Euler's formula (Eq. (1)) relating a lower bound on the number of edges around a face to the average vertex degree.
3. Sec. 6 derives a lower bound of 4 on the number of face edges with nonorthogonal dihedral angle in a subgraph of the 1-skeleton of the polyhedron, and uses this to establish the claim by contradiction.

4 Local Vertex Constraints

We now embark on a study of the possible configurations of rectangles glued to one vertex of degree δ. Each rectangle incident to a vertex glues a face angle of $\pi/2$ there. It is possible for an arbitrary number δ of such angles to be glued to one vertex: they can squeeze together like an accordion. What is not possible is for there to be, in addition, an arbitrary number of dihedral angles that are multiples of $\pi/2$ incident to one vertex. It is this tradeoff we explore here.

4.1 Vertices with k Red Edges

We employ preliminary lemmas in the full version to derive constraints on vertices with few incident red edges (edges at which the dihedral angle is not a multiple of $\pi/2$). We will show that it is not possible to have just one, or three incident red edges, that two and four red edges force collinearities, and that zero or five (or more) red edges are possible. A summary is listed below.

k	Constraint	Lemma
0	possible	
1	not possible	Lem. 6
2	collinear	Lem. 7
3	not possible	Lem. 8
4	'+' shape	Lem. 9
5	possible	

Lemma 9 *If a vertex has exactly four incident red edges, then they must fall into two collinear pairs meeting orthogonally, forming a '+' in 3-space.*

It is possible to have five red edges incident to a vertex. Fig. 17 shows one such example, where a quarter pair (p_1, p_5) is connected by four arcs, connecting five edges none of whose dihedral angles are rectilinear. It is also possible to have more than five red edges: any larger number can accordion-fold into a small space, say connecting a quarter pair of points.

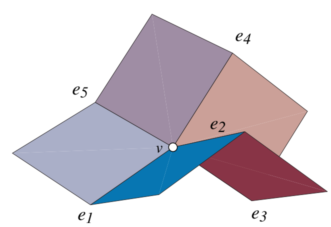

Figure 17: Five red edges $e_1, \ldots, e_5$ incident to v.

6 Orthogonality Forced for Genus Zero and One

We establish in this section that the answer to Question 2 is YES for polyhedra of genus zero and one. We first define a red subgraph, then prove that $k = 4$ for it, and, finally, exploit corollaries from the previous section.

6.1 Red Subgraph G_r

Starting with the 1-skeleton G of the polyhedron, with its edges colored green or red depending on whether the dihedral angle is rectilinear or not, we perform the following operations to reach G_r:

1. Remove all green edges from G, retaining only red edges.
2. Merge edges meeting at a degree-2 vertex: if a node y is of degree 2, with incident edges (x, y) and (y, z), delete y and those edges and replace with (x, z).
3. Select one component of the resulting graph and call it G_r, the *red subgraph*.

If G contains any red edge, then there is a nonempty G_r. Note that G_r is realized in $\mathbb{R}^3$ with straight segments for each edge, because Lem. 7 guarantees that the "erasing" of degree-2 vertices merges collinear polyhedron edges.

6.2 Facial Walks in G_r

G_r is naturally embedded on the surface of the polyhedron. Call this its *canonical embedding*.

Lemma 13 *Every face walk of the canonical embedding of G_r contains at least four edges.*

Four edges are needed to close a cycle; four right angles force the cycle edges to lie in a plane, and therefore form a rectangle.

6.3 Concluding Theorem

Theorem 14 *Any polyhedron of genus zero or one, all of whose faces are rectangles, must be an orthogonal polyhedron: all of its dihedral angles are multiples of $\pi/2$.*

One way to view the proof of this theorem is that a red cycle cannot turn in $\mathbb{R}^3$ with only vertices of red-degree no more than 4. It requires degree-5 vertices, or higher degree vertices, to permit a cycle to form. This is exactly how the polyhedron described in Sec. 2 is constructed. The four corners of each square in Fig. 2 are red-degree 5 vertices, and the fifth vertex $\frac{1}{2}$ offset from the center of each square is red-degree 8. It seems the computations used in the previous theorem provide no useful constraint when $g \geq 2$.

7 Discussion

The obvious open problem is to close the gap—between Theorem 15, $g \leq 1$, when orthogonality in $\mathbb{R}^3$ is forced, and Theorem 1, $g \geq 7$, when it is not forced. As mentioned in the Introduction, this gap has been narrowed by the Waterloo group to $g \leq 2$ forcing orthogonality, and $g \geq 6$ not forced. Is there a nonorthogonal polyhedron of genus $g \in \{3, 4, 5\}$ constructed entirely from rectangles?

Perhaps a more interesting problem is to extend these results to other planar constraints, and ask if they imply a restriction in $\mathbb{R}^3$. For example, if a polyhedron is constructed out of convex polygons whose angles are all multiples of π/k, $k > 2$, does this imply any restriction on the realizable dihedral angles?

Finally, because so little is known about nonoverlapping (simple) edge-unfoldings, perhaps just-barely nonsimple unfoldings should be considered.

Acknowledgments We thank Michael Albertson for advice on graph embeddings, Erik Demaine for useful comments, Paul Nijjar for finding an error in an earlier version of Lemma 13 (formerly Lemma 14), and Timothy Chan for finding an error in a (now removed) figure.

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