Geometric Permutations of Balls with Bounded Size Disparity

Yunhong Zhou
Compaq System Research Center
130 Lytton Ave
Palo Alto, CA 94301
yzhou@pa.dec.com

Subhash Suri
Computer Science Department
University of California
Santa Barbara, CA 93106
suri@cs.ucsb.edu

Abstract

We study combinatorial bounds for geometric permutations of balls with bounded size disparity in d-space. Our main contribution is the following theorem: given a set $\mathcal{S}$ of n disjoint balls in $\mathcal{R}^d$, if n is sufficiently large and the radius ratio between the largest and smallest balls of $\mathcal{S}$ is γ, then the maximum number of geometric permutations of $\mathcal{S}$ is $O(\gamma^{\log \gamma})$. When $d = 2$, we are able to prove the tight bound of 2 on the number of geometric permutations for $\mathcal{S}$, which is the best possible bound because it holds even when $\gamma = 1$. Our theorem shows how the number of permutations varies as a function of the size disparity among balls, thus gracefully bridging the gap between two extreme bounds that were known before: the $O(1)$ bound for congruent balls, and the $\Theta(n^{d-1})$ bound for arbitrary balls.

1 Introduction

A line ℓ is called a *line transversal* for a set $\mathcal{S}$ of bodies in $\mathcal{R}^d$ if it intersects every member of $\mathcal{S}$. If the objects in $\mathcal{S}$ are pairwise disjoint convex bodies, then a line transversal of $\mathcal{S}$ induces two linear orderings, which are the orders in which ℓ meets the members of $\mathcal{S}$. The two orders induced by a line transversal are essentially the same (one is reverse of the other), and so they are together called a single *geometric permutation*. Let $g_d(n)$ denote the maximum number of geometric permutations, where the maximum is taken over all such families $\mathcal{S}$ of size n in $\mathcal{R}^d$. Then, the following asymptotic bounds on $g_d(n)$ have been known for some time:

1. $g_2(n) = 2n - 2$ [2].
2. $g_d(n) = \Omega(n^{d-1})$ [4].
3. $g_d(n) = O(n^{2d-2})$ [7].

Thus, for dimensions $d \geq 3$, there is a substantial gap between the known upper and lower bounds on geometric permutations. This gap motivated Smorodinsky, Mitchell, and Sharir [5] to consider a natural special case of convex bodies: *balls*. They proved that the maximum number of geometric permutations achieved by n pairwise disjoint balls in $\mathcal{R}^d$ is $O(n^{d-1})$. Moreover, this bound is tight as they were able to show a matching lower bound construction.

An interesting special case of the problem is when all the balls in $\mathcal{S}$ are congruent (say, unit radius balls). It is conjectured that for unit radius balls, the number of geometric permutations drops precipitously to $O(1)$, with the constant depending on d. The *planar* case of this conjecture was proved by Katchalski and Asinowski [1] and also by Smorodinsky et al. [5]. They showed that

when $|\mathcal{S}|$ is sufficiently large, the maximum number of geometric permutations admitted by n unit disks is *two*.

Recently, we proved the unit radius ball conjecture for arbitrary dimensions, and showed that for n congruent balls in $\mathcal{R}^d$, the maximum number of geometric permutations is *four* [8]. Here n, the size of the set $|\mathcal{S}|$, depends on the dimension constant d. However, as opposed to the conjecture made by Smorodinsky, Mitchell, and Sharir [5], the bound on geometric permutations is *independent* of d.

The sharp difference between the bounds for congruent and arbitrary balls is quite intriguing. The bound jumps abruptly from four to $\Theta(n^{d-1})$ when the congruency constraint is removed. For instance, even in the case of planar disks, the number of permutations is at most 2 if the disks are congruent, but can be $\Theta(n)$ otherwise. A closer examination of the problem reveals that constructions that achieve $\Omega(n)$ distinct permutations for disks require *exponentially varying* radii, and the same phenomenon occurs for higher dimensions as well. Thus, both from a practical as well as a mathematical perspective, it is an open question as to how the number of permutations grows as a function of the varying radii. In particular, is the number of permutations a constant if the ratio between the largest and smallest ball is bounded by a constant? Our earlier proof for the unit ball case depended heavily on the congruency of balls, and didn't shed any light on the geometric permutations of non-congruent balls.

1.1 Our Contributions

In this paper, we investigate the geometric permutation problem for disjoint non-congruent balls in d dimensions, $d \geq 2$. Our bound depends on the radius ratio between the largest and the smallest balls. Specifically, assume that $\mathcal{S} = \{B_1, \cdots, B_n\}$ is a set of disjoint balls in $\mathcal{R}^d$, where $B_i \equiv B(o_i, r_i)$, meaning that B_i has center o_i and radius r_i. Define

$$\gamma(\mathcal{S}) \;=\; \max_{i,j} \frac{r_i}{r_j}.$$

We call $\gamma(\mathcal{S})$ the *radial ratio* of $\mathcal{S}$.

Our main result is the following: Let $n = |\mathcal{S}|$ and $\gamma = \gamma(\mathcal{S})$. If n is sufficiently large[1], then the maximum number of geometric permutations of $\mathcal{S}$ is $O(\gamma^{\log \gamma})$. When $d = 2$, we are able to prove the tight bound of 2 on the number of geometric permutations for $\mathcal{S}$, which is the best possible bound because it holds even when $\gamma = 1$.

Our bounds provide a more refined analysis of the geometric permutation problem, showing how the number of permutations varies as a function of the size disparity among balls, and gracefully bridging the gap between two extreme bounds that were known before: the $O(1)$ bound for congruent balls, and the $\Theta(n^{d-1})$ bound for arbitrary balls.

A simple, but quite practical, corollary of our result is the following. Let $\mathcal{S}$ be a set n pairwise disjoint convex bodies in d-space such that their *smallest enclosing balls* are also pairwise disjoint. If the radial ratio of the enclosing balls is bounded by γ, then the number of geometric permutations for $\mathcal{S}$ is $O(\gamma^{\log \gamma})$. In other words, even arbitrary convex bodies admit only a constant number of permutations *if* they are relatively well-separated and have bounded size disparity. By contrast, without these assumptions, the best upper bound known for permutations of convex objects is $O(n^{2d-2})$, and there is a lower bound of $\Omega(n^{d-1})$.

While in combinatorial geometry line transversals and geometric permutations have been studied primarily for their mathematical appeal, they also find practical applications in computing visibility

[1] Here we mean that n is asymptotically greater than a polynomial of γ, d is a constant. Details about this bound can be found in Theorem 4.5.

information in 3-dimensional computer games and architectural walkthroughs. These applications decompose the space into boxes, and then try to compute visibility between various boxes. The openings through which light can pass, such as windows, doors and the tops of stairwells in a building model, are modeled by convex *portals* on the faces of these boxes; all other objects are considered opaque. The visibility among boxes is computed from the visibility between these portals. See Teller and Sequin [6] for details. Similar ideas are used in various video games as well.

Due to lack of space, we omit details of several key proofs from the main abstract, but include them in the appendix.

2 Properties of Line Transversals

In this section, we establish a series of properties for line transversals on a set of disjoint balls with bounded size disparity. These properties are cornerstones for us to bound the number of geometric permutations in subsequent sections.

Let $\mathcal{S} = \{B_1, \cdots, B_n\}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$, and let $\gamma = \gamma(\mathcal{S})$ be the radial ratio of $\mathcal{S}$. We use the notation $B_i = B(o_i, r_i)$ to signify that the center of B_i is o_i and its radius is r_i, for $1 \leq i \leq n$. With proper scaling, we can assume that $1 \leq r_i \leq \gamma$, for $1 \leq i \leq n$. Our analysis relies on the critical assumption that n is large enough, and that γ is fixed. We will slightly abuse the notation and write *diameter* of $\mathcal{S}$ to mean the diameter of the set of centers of all the balls in $\mathcal{S}$. That is, $D(\mathcal{S}) = \max_{ij} \operatorname{dist}(o_i, o_j)$.

Lemma 2.1 *Let $\mathcal{S}$ be a set of n disjoint balls in $\mathcal{R}^d$ with radial ratio γ. If $\mathcal{S}$ admits at least one geometric permutation and $n = \omega(\gamma^d)$,[2] then the diameter of $\mathcal{S}$ is $\Omega(n/\gamma^{d-1})$.*

Proof. Let ℓ be a line transversal of $\mathcal{S}$. The projection of $\mathcal{S}$ onto ℓ has length at most $2\gamma + D$, here D is the diameter of $\mathcal{S}$. Thus all members of $\mathcal{S}$ are contained within a radius 2γ cylinder of height $2\gamma + D$. If V_d denotes the volume of a unit ball in $\mathcal{R}^d$, then the volume of this cylinder is $(2\gamma + D)V_{d-1}(2\gamma)^{d-1}$. Since n disjoint balls are contained in this cylinder, and d is a constant we get

$$(2\gamma + D)V_{d-1}(2\gamma)^{d-1} \geq nV_d, \text{ which implies that}$$
$$D \geq \frac{V_d}{2^{d-1}V_{d-1}} \frac{n}{\gamma^{d-1}} - 2\gamma = \Omega\left(\frac{n - \gamma^d}{\gamma^{d-1}}\right) = \Omega\left(\frac{n}{\gamma^{d-1}}\right).$$

The last inequality uses the fact that $n = \omega(\gamma^d)$. □

Lemma 2.2 *Let $B_1 = B(o_1, r_1)$ and $B_2 = B(o_2, r_2)$ be two disjoint balls in $\mathcal{R}^d$, and let ℓ be a line that intersects them both. Let $L = \operatorname{dist}(o_1, o_2)$ and let θ denote the angle between ℓ and the line containing o_1, o_2. Then, we have $\sin\theta \leq \frac{r_1 + r_2}{L}$, and θ achieves its maximum value θ_0 when ℓ is a common tangent of B_1, B_2 passing through their geometric center.*

Lemma 2.3 *Suppose $\mathcal{S}$ is a set of n disjoint balls in $\mathcal{R}^d$ with radial ratio γ, there is at least one geometric permutation of $\mathcal{S}$, and $n = \omega(\gamma^d)$. Let z-axis be the line connecting two centers with the longest distance. Then the angle between any line transversal of $\mathcal{S}$ and z-axis is bounded by $O(\gamma^d/n)$; consequently, the angle between any two line transversals of $\mathcal{S}$ is also $O(\gamma^d/n)$.*

[2] Recall that $f(n) = \omega(g(n)) \iff \lim_{n\to\infty} f(n)/g(n) = \infty$.

PROOF. Let $B(o_1, r_1), B(o_2, r_2)$ be two balls with the longest distance between their centers. Thus z-axis connects o_1 and o_2 and $D = \text{dist}(o_1, o_2)$ is the diameter of $\mathcal{S}$. By Lemma 2.1, $D = \Omega(n/\gamma^{d-1})$. Given any line transversal ℓ, if θ is the angle between ℓ and z, by Lemma 2.2 we have

$$\theta \;\leq\; \frac{\pi}{2}\sin\theta \;\leq\; \frac{\pi}{2}\,\frac{r_1 + r_2}{D} \;\leq\; \pi\,\frac{\gamma}{D} \;=\; O\left(\frac{\gamma^d}{n}\right).$$

The preceding inequalites have used the following fact: $\theta \leq \frac{\pi}{2}\sin\theta$. This is true because θ is the angle between two lines, thus in the range $[0, \pi/2]$. The angle between any two line transversals of $\mathcal{S}$ is at most twice the maximal angle between one line transversal and the z-axis, and it should also be bounded by $O(\gamma^d/n)$. □

With the assumption $n = \omega(\gamma^d)$, the angle between any two line transversals is very small, approaching zero when n is large enough. In the following, when we talk about line transversals, we will *always* mean directed lines passing through all the balls, with their directions nearly the same as $\vec{z}$, the first coordinate axis of the system.

2.1 One Technical Lemma

Given two disjoint balls $B(o_1, r_1)$ and $B(o_2, r_2)$, their geometric center is defined as $(r_2 o_1 + r_1 o_2)/(r_1 + r_2)$. In the plane, the geometric center of two disjoint disks is the intersection point of their common inner tangents. The geometric center is collinear with the centers of these two balls, and lies outside of both balls. In order to bound geometric permutations quantitatively, we need a technical lemma to estimate the *closeness* of line transversals to geometric centers. Lemma 2.6 is such a key technical lemma, and it establishes an important algebraic condition for the line transversal of two balls. Intuitively, the lemma shows that if two line transversals intersect two balls in different orders, then these two lines must pass *very close* to the geometric center of these two balls; the closeness depends on the radii of the balls. The proof, which is included in the appendix, rests on the following two simpler lemmas.

Lemma 2.4 *Let $B(o, r)$ be a ball in $\mathcal{R}^d$ with center $o = (o_1, o_2, \cdots, o_d)$ and radius r. Let $\ell = \{\ell(t) \mid t \in \mathcal{R}\}$ be the parametric equation of a line with $\ell(t) = (t, at + b, c_3, \cdots, c_d)$. If ℓ has a nonempty intersection with B, then $\ell(t_0) \in B$, where $t_0 = \frac{o_1 + a(o_2 - b)}{a^2 + 1}$, or equivalently*

$$\frac{(ao_1 - (o_2 - b))^2}{1 + a^2} + \sum_{3 \leq i \leq d} (c_i - o_i)^2 \;\leq\; r^2 \;\;.$$

Lemma 2.5 *Let $\lambda_1, \lambda_2 > 0$ and $a, b \in R$. Then the following inequality holds:*

$$\lambda_1(x - a)^2 + \lambda_2(x - b)^2 \;\geq\; \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}(a - b)^2, \quad \forall\, x \in R.$$

Lemma 2.6 *Let $P = B(p, r_p)$ and $Q = B(q, r_q)$ be two disjoint balls in $\mathcal{R}^d$ with centers p, q and radii r_p, r_q respectively. Assume that their geometric center has its first coordinate equal to zero, and that the first coordinate axis $\vec{z}$ intersects both P and Q, with P first. If another oriented line ℓ with parametric form $\{\ell(t) = (t, at + b, c_3, \cdots, c_d) \mid t \in \mathcal{R}\}$ intersects both P and Q, with Q first, then we must have $|b| \leq \sqrt{r_p r_q}\, a$. Here $a > 0$ is the slope of line ℓ, and $c_3, \cdots, c_d$ are all constants.*

2.2 Switched Pairs

We say that a pair of balls B_1, B_2 forms a *switched pair* in $\mathcal{S}$ if there exist two line transversals such that they meet B_1, B_2 in different orders. Recall that line transversals are oriented lines with almost the same direction. Switched pairs are central to geometric permutations, and our next few lemmas establish key properties that allow us to bound their number. The first property shows that two balls making a switched pair are very close.

Lemma 2.7 *The distance between two balls making a switched pair is* $O(\gamma^{2d+1}/n^2)$.

Proof. Let $\mathcal{S}$ be a set of disjoint balls in $\mathcal{R}^d$ with radial ratio γ. (B_1, B_2) is a switched pair of $\mathcal{S}$ with $B_1 = B(o_1, r_1), B_2 = B(o_2, r_2)$. Denote $\delta = \text{dist}(B_1, B_2)$ and $L = \text{dist}(o_1, o_2)$, thus $L = r_1 + r_2 + \delta$. From the definition of switched pairs, thus there exist two line transversals ℓ and ℓ' such that they cut B_1 and B_2 in different orders. Let 2ϕ be the angle between ℓ and ℓ'. Furthermore, let θ, θ' denote respectively the angles between $\overrightarrow{o_1 o_2}$ and ℓ, ℓ'. See Figure 1.

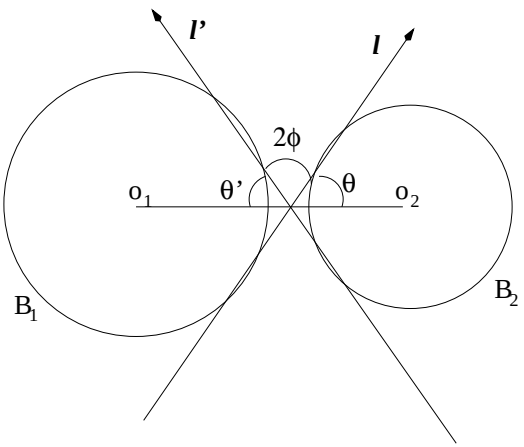

Figure 1: An illustration of two balls B_1, B_2 making a switched pair with $d = 2$.

From Lemma 2.3, we have $\phi = O(\gamma^d/n)$. Recalling from Lemma 2.2 that θ_0 is the tangency angle, we get $\theta \le \theta_0, \theta' \le \theta_0$, and

$$\sin\theta_0 = \frac{r_1 + r_2}{L} = \frac{r_1 + r_2}{r_1 + r_2 + \delta}, \quad \text{which implies that}$$
$$\delta \;=\; (r_1 + r_2)\left(\frac{1}{\sin\theta_0} - 1\right) \;\le\; 2\gamma\left(\frac{1}{\sin\theta_0} - 1\right).$$

Because ℓ, ℓ' cut two balls in different orders, we can check from standard geometry that $\theta + \theta' + 2\phi \ge \pi$. Figure 1 gives an illustration for the plane case, and in this case equality holds. Therefore

$$2\theta_0 + 2\phi \ge \theta + \theta' + 2\phi \ge \pi \quad \Longrightarrow \quad \theta_0 \ge \frac{\pi}{2} - \phi \quad \Longrightarrow \quad \sin\theta_0 \ge \cos\phi.$$

By combining the above two inequalities, we get

$$\begin{aligned} \delta \;&\le\; 2\gamma\left(\frac{1}{\sin\theta_0} - 1\right) \;\le\; 2\gamma\left(\frac{1}{\cos\phi} - 1\right) \\ &=\; 2\gamma\,\frac{2\sin^2\frac{\phi}{2}}{\cos\phi} \;\le\; \gamma\,\frac{\phi^2}{\cos\phi} \end{aligned}$$

$$= \quad O\left(\gamma\ \phi^2\right)\ =\ O\left(\frac{\gamma^{2d+1}}{n^2}\right).$$

In the last inequality we used the fact $\phi = O(\gamma^d/n)$. This completes the proof. □

Lemma 2.8 *Let (B_1, B_2) be a switched pair in $\mathcal{S}$, and let o_1, o_2 be the centers of these balls. If ℓ is a line transversal of $\mathcal{S}$, then the line ℓ is nearly perpendicular to the line passing through o_1, o_2. More specifically, the angle between ℓ and $\overrightarrow{o_1 o_2}$ is $\pi/2 - O(\gamma^d/n)$.*

Before we introduce the unique pairing property for general balls, we need one more technical lemma.

Lemma 2.9 *Let $B = B(o, r), B_1 = B(o_1, r_1), B_2 = B(o_2, r_2)$ be three balls in $\mathcal{R}^d$ with $\mathrm{dist}(B, B_1) = 0, \mathrm{dist}(B, B_2) = 0, \mathrm{dist}(B_1, B_2) \geq 0$, where r, r_1, r_2 are all positive numbers greater than or equal to 1. If E is the geometric center of $\{B, B_1\}$, and F is the geometric center of $\{B, B_2\}$, then $\mathrm{dist}(E, F) \geq 1$.*

The following lemma describes one of the most important properties of switched pairs: *unique pairing property.*

Lemma 2.10 *Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. $\mathcal{S}$ has at least one line transversal with $n = \omega(\gamma^{d+1/2})$. Then, a ball of $\mathcal{S}$ can appear in at most one switched pair.*

The unique pairing property of switched pairs easily implies the fact that if there are m switched pairs, then there can be at most 2^m permutations—each permutation is defined by one of the two orders of some switched pairs.

Lemma 2.11 *Given a set $\mathcal{S}$ of n disjoint convex objects in $\mathcal{R}^d$. If the unique pairing property holds for switched pairs, then two members making a switched pair should always appear consecutively in all line transversals of $\mathcal{S}$. And if there are a total of m switched pairs, then $\mathcal{S}$ admits at most 2^m different geometric permutations.*

Proof. We prove the first statement by contradiction. Suppose that $\{P, Q\}$ is a switched pair of $\mathcal{S}$, and in one line transversal ℓ the order is $(\cdots, P, \cdots, X, \cdots, Q, \cdots)$ with another object X between P and Q. Because $\{P, Q\}$ is a switched pair, there must exist another line transversal ℓ' which meets Q before P. There are three possible positions for X to meet ℓ'. If the order is either $(\cdots, X, \cdots, Q, \cdots, P, \cdots)$ or $(\cdots, Q, \cdots, X, \cdots, P, \cdots)$, then $\{P, X\}$ is also a switched pair, which violates the unique pairing property for P. If the order is $(\cdots, Q, \cdots, P, \cdots, X, \cdots)$, then $\{X, Q\}$ is also a switched pair, which violates the unique pairing property for Q. Then, if two objects form a switched pair, they should always appear consecutively in any geometric permutation.

Based on the first statement, the second statement is trivial. If we can consider two objects in one switched pair as just one object, then the order for all objects to appear in any line transversal is fixed. For each switched pair, there are two possible orders for those two objects to appear in any line transversal, so that in total there should be at most 2^m different orders. □

The following important fact is easily obtained by combining Lemma 2.10 and Lemma 2.11.

Lemma 2.12 *Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. If $n = \omega(\gamma^{d+1/2})$, then the unique pairing property of switched pairs holds. If $\mathcal{S}$ contains m switched pairs, then $\mathcal{S}$ admits at most 2^m geometric permutations.*

3 Bounds for the Planar Case

In this section, we focus on the two-dimensional case of geometric permutations of non-congruent disks. The general-dimensional case is more complex, and is handled separately in the next section. We start with the following fact, which is a restatement of Lemma 2.10 for $d = 2$.

Corollary 3.1 *For a set $\mathcal{S}$ of n disjoint disks in $\mathcal{R}^2$, every disk can appear in at most one switched pair assuming $n = \omega(\gamma^{5/2})$.*

The next lemma states a fact proved in [3]. Please note that this fact only holds in the plane, which is also the reason that we need to treat the cases $d = 2$ and $d \geq 3$ separately.

Lemma 3.2 *Let $\mathcal{S} = \{a, b, c, d\}$ be a set of four pairwise disjoint convex bodies in the plane. If (a, b, c, d) is a geometric permutation of $\mathcal{S}$, then (b, a, d, c) cannot be a geometric permutation of $\mathcal{S}$.*

Lemma 3.3 *Consider two switched pairs: $P = \{B_1, B_2\}$ and $P' = \{B_1', B_2'\}$. Then, there must exist two line transversals ℓ and ℓ' that meet both switched pairs in different orders. In other words, there exist line transversals ℓ and ℓ' such that ℓ meets B_1 before B_2 and B_1' before B_2', while ℓ' meets B_2 before B_1 and B_2' before B_1'.*

Our main result is the following theorem.

Theorem 3.4 *[8] Let $\mathcal{S}$ be a set of n disjoint balls in $\mathcal{R}^2$ with radial ratio $\gamma = \gamma(\mathcal{S})$. If $n = \omega(\gamma^{5/2})$, then there are at most two geometric permutations on $\mathcal{S}$.*

Proof. By Corollary 3.1, the unique pairing property for switched pairs holds here. Based on Lemma 2.12, if $\mathcal{S}$ contains m switched pairs, then there should be at most 2^m geometric permutations. Based on Lemma 3.2 and Lemma 3.3, it follows immediately that we cannot have two switched pairs. So that $m \leq 1$. Thus, the number of geometric permutations is at most two. □

4 Bounds for Higher Dimensions

We are now ready to prove our main theorem, for any fixed dimension d. Recall that $\mathcal{S} = \{B_1, \cdots, B_n\}$ is a set of disjoint balls in $\mathcal{R}^d$, whose radial ratio is $\gamma = \gamma(\mathcal{S})$. Ball B_i has center o_i and radius r_i, where we assume without loss of generality that $1 \leq r_i \leq \gamma$, for $1 \leq i \leq n$.

We transform the coordinate system so that z-axis is oriented along the line connecting two centers with the longest distance in $\mathcal{S}$. For each switched pair, we project its geometric center onto the z-axis and call the projected coordinate the *height* of this switched pair. Thus, all the switched pairs of $\mathcal{S}$ can be linearly ordered using their heights. The key components of our proof are lower and upper bounds on the height distance between geometric centers of two switched pairs. The next two lemmas establish these. (Please see the appendix for their proofs.)

Lemma 4.1 *Let $\{B_1, B_1'\}$ and $\{B_2, B_2'\}$ be two switched pairs of $\mathcal{S}$ with centers o_1, o_1', o_2, o_2' and radii r_1, r_1', r_2, r_2' respectively. The height distance between the geometric centers of these two switched pairs is at least $\sqrt{\frac{4r_2\ r_1\ r_1'}{r_1 + r_1'}} - \epsilon(n)$, where $\lim_{n\to\infty} \epsilon(n) = 0$.*

Lemma 4.2 *Let $\{B_1, B_1'\}$ and $\{B_2, B_2'\}$ be two switched pairs of $\mathcal{S}$ with centers o_1, o_1', o_2, o_2' and radii r_1, r_1', r_2, r_2' respectively. The height distance between the geometric centers of these two switched pairs is at most $\sqrt{r_1\ r_1'} + \sqrt{r_2\ r_2'} + \epsilon(n)$, where $\lim_{n\to\infty} \epsilon(n) = 0$.*

Theorem 4.3 *Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. If $d \geq 3$ is a constant and $n = \omega(\gamma^{d+1/2})$, then the total number of switched pairs is bounded by $1 + \lfloor\sqrt{2}\gamma\rfloor$. Consequently, $\mathcal{S}$ admits at most $2^{1+\lfloor\sqrt{2}\gamma\rfloor}$ geometric permutations.*

PROOF. Suppose that there are totally m switched pairs. We sort them based on their *heights*, which is defined at the beginning of this section. Let $SP_1, SP_2, \cdots, SP_m$ denote the sorted order of these switched pairs. Here $SP_i = \{B(o_i, r_i), B(o_i', r_i')\}$ for $1 \leq i \leq m$. From Lemma 4.2, the height difference between geometric centers of SP_1, SP_m is bounded by

$$\sqrt{r_0 r_0'} + \sqrt{r_m r_m'} + \epsilon(n) \ \leq \ 2\gamma + \epsilon(n).$$

By Lemma 4.1, for two adjacent switched pairs SP_{i-1}, SP_i, their geometric centers have height difference lower bounded by

$$\sqrt{\frac{4 r_i r_{i-1} r_{i-1}'}{r_{i-1} + r_{i-1}'}} - \epsilon'(n) \geq \sqrt{2} - \epsilon'(n).$$

Since m is the total number of switched pairs, we have

$$m - 1 \leq \frac{2\gamma + \epsilon(n)}{\sqrt{2} - \epsilon'(n)} \leq \sqrt{2}\gamma + \epsilon''(n).$$

Because m is an integer, and $\lim_{n\to\infty} \epsilon''(n) = 0$, we must have $m \leq 1 + \lfloor\sqrt{2}\gamma\rfloor$ when n is sufficiently large. Thus, by Lemma 2.12, $\mathcal{S}$ admits at most $2^{1+\lfloor\sqrt{2}\gamma\rfloor}$ geometric permutations. □

The above theorem gives an upper bound of 4 for unit balls when $\gamma = 1$, thus generalizing our result for congruent balls [8]. While this bound is fairly tight when γ is small, its exponential dependence on γ makes it less attractive as γ gets larger. In the following, we improve this bound substantially using Hamming distance to argue about distinct geometric permutations.

4.1 Permutation Bounds Using Hamming Distances

Let $H^d = \{(x_1, \cdots, x_d) \mid x_i \in \{0, 1\},\ 1 \leq i \leq d\}$ denote the d-dimensional Hamming space. For any $x, y \in H^d$, $\text{dist}_H(x, y)$ is the Hamming distance between x and y. Recall that the Hamming distance is the number of different coordinates between x and y. In the following, we will use Hamming numbers to represent geometric permutations, and bound the total number of geometric permutations by using properties of Hamming space.

Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. Assume that $d \geq 3$ is a constant and $n = \omega(\gamma^{d+1/2})$. We can order all the switched pairs by their heights. Let $SP_1, SP_2, \cdots, SP_m$ be this sorted order. We characterize each geometric permutation by the order in which it intersects all the switched pairs. Let $SP_i = \{B(o_i, r_i), B(o_i', r_i')\}$, and assume that $r_i \geq r_i'$ for $1 \leq i \leq m$. For each geometric permutation, we can map it to a Hamming number $x = (x_1, \cdots, x_m)$. If $B(o_i, r_i)$ appears before $B(o_i', r_i')$ in the permutation, then $x_i = 1$. Otherwise $x_i = 0$. Now, we can prove the following fact.

Lemma 4.4 *Let $x, y \in H^m$ denote the representations of two geometric permutations of $\mathcal{S}$. Then $\text{dist}_H(x, y) \leq 2(1 + \lfloor\log\gamma\rfloor)$.*

With the above lemma, we are ready to prove our major result, which improves the upper bound in Theorem 4.3.

Theorem 4.5 *Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. If $d \geq 3$ is a constant and $n = \omega(\gamma^{d+1/2})$, then $\mathcal{S}$ admits $O\left(\gamma^{\log \gamma}\right)$ geometric permutations.*

PROOF. Let $SP_1, SP_2, \cdots, SP_m$ be the list of all switched pairs in sorted order. From Theorem 4.3,

$$m \leq 1 + \lfloor \sqrt{2}\gamma \rfloor.$$

By Lemma 4.4, for any two geometric permutations represented by Hamming numbers in H^m, their distance is upper bounded by

$$k = 2(1 + \lfloor \log \gamma \rfloor).$$

Let $A(d, j)$ denote the maximal cardinality of subsets of H^d with elements at distance $\leq j$. Then the total number of geometric permutations is bounded by $A(m, k)$. In the following we derive our estimate of $A(m, k)$.

It is easy to realize that $A(m, k)$ is increasing with respect to m. So that we can assume that $m = 1 + \lfloor \sqrt{2}\gamma \rfloor$. (Note that $k = 2(1 + \lfloor \log \gamma \rfloor)$ is much smaller than m.) Let $j = \lfloor k/2 \rfloor = 1 + \lfloor \log \gamma \rfloor$. First we claim the following inequality:

$$A(m, k) \ \leq \ 2 + (2^{k-1} - 1) \sum_{i=0}^{j} \binom{m-k}{i}. \tag{1}$$

Let $A \subset H^m$ denote a subset of H^m realizing maximal cardinality with diameter bounded by k. It is easy to see that A should have diameter k, or equivalently there exist two elements of A with distance k. We can write these two elements as $x = (1, \cdots, 1)$ and $y = (0, \cdots, 0, 1, \cdots, 1)$, where y has its first k coordinates 0. Now for any other element $z \in A$, we split it into $z = (z_1, z_2)$, where z_1 is the first k coordinates, and z_2 is the remaining $m - k$ coordinates. If z_1 has all coordinates 1 or all coordinates 0, then z_2 should have all coordinates 1, and then z coincides with x or y. So that z_1 has only $2^k - 2$ choices. For a fixed z_1, either it has at least $\lceil k/2 \rceil$ coordinates different from x, or it has at least $\lceil k/2 \rceil$ coordinates different from y. Thus the distance from z_2 to all 1's should be at most $\lfloor k/2 \rfloor$. Equivalently, z_2 has at most $\lfloor k/2 \rfloor$ zeros. Therefore the total number of choices for z_2 is bounded by $\sum_{i=0}^{j} \binom{m-k}{i}$. Furthermore, we can group these $2^k - 2$ choices of z_1 into conjugate pairs. (We call two elements conjugate if their corresponding coordinates are all different.) It can be shown that for every conjugate pair, the total number of elements is still bounded by $\sum_{i=0}^{j} \binom{m-k}{i}$. So that (1) is proved.

Eq. (1) implies that

$$A(m, k) \ \leq \ 2^{k-1} \sum_{i=0}^{j} \frac{(m-k)^i}{i!} \ = \ O\left(\frac{2^k \ m^j}{j!}\right). \tag{2}$$

We also have

$$\begin{aligned} 2^k &= 2^{2(1+\lfloor \log \gamma \rfloor)} = O(2^{2 \log \gamma}) = O(\gamma^2), \\ m^j &= (1 + \lfloor \sqrt{2}\ \gamma \rfloor)^{1+\lfloor \log \gamma \rfloor} = O\left((\sqrt{2}\ \gamma)^{1+\log \gamma}\right) = O\left(\gamma^{3/2+\log \gamma}\right), \\ j! &= \Omega\left((j/e)^j\right) = \Omega\left((\log \gamma / e)^{\log \gamma}\right) = \Omega\left(\gamma^{\log \log \gamma - \log e}\right). \end{aligned}$$

By plugging the expressions of the above three equations into Eq. (2), we get

$$A(m, k) = O\left(\gamma^{\log \gamma + O(1) - \log \log \gamma}\right) = O\left(\gamma^{\log \gamma}\right).$$

The exact constant in the "big Oh" is $7/2 + \log e$. This completes the proof. □

5 Conclusion

We considered the problem of geometric permutations for non-congruent balls in d dimensions, and offered a substantially refined and shape-sensitive analysis. The previous results for this problem had shown worst-case optimal bounds for two extreme cases of the input variety: for congruent balls, the number of distinct permutations is $O(1)$, while for arbitrary balls, the number of permutations is $\Theta(n^{d-1})$. We considered the natural question: how does the combinatorial complexity of geometric permutations increase as we relax the congruency constraint. Our best upper bound is $O(\gamma^{\log \gamma})$, where γ is radius ratio between the largest and smallest balls of $\mathcal{S}$. In particular, for constant γ, the number of permutations is also a constant.

In the planar case, we were able to prove the tight bound of 2. Thus, for any set of n disks with radius ratio γ and $n = \omega(\gamma^{5/2})$, the number of permutations is at most 2.

The idea of incorporating shape-sensitivity in the analysis of geometric algorithms seems to be a powerful and fruitful one. There appear to be many cases where the bounds obtained through the traditional analysis are quite pessimistic—the constructions creating lower bounds are highly pathological and involve exponential shape or size disparity, which may be extremely unlikely in practice. A more refined analysis, like the one offered in this paper, can provide a more useful insight into the problem complexity.

References

[1] A. Asinowski. Common transversals and geometric permutations. Master's thesis, Technion IIT, Haifa, Israel, 1998.

[2] H. Edelsbrunner and M. Sharir. The maximum number of ways to stab n convex non-intersecting sets in the plane is $2n - 2$. *Discrete Computational Geometry*, 5(1):35–42, 1990.

[3] M. Katchalski, T. Lewis, and A. Liu. Geometric permutations of disjoint translates of convex sets. *Discrete Math.*, 65:249–259, 1987.

[4] M. Katchalski, T. Lewis, and A. Liu. The different ways of stabbing disjoint convex sets. *Discrete Comput. Geom.*, 7:197–206, 1992.

[5] Shakhar Smorodinsky, Joseph S.B. Mitchell, and Micha Sharir. Sharp bounds on geometric permutations for pairwise disjoint balls in R^d. *Discrete and Computational Geometry*, 23:247–259, 2000.

[6] S. Teller and C. Sequin. Visibility preprocessing for interactive walkthroughs. In *Computer Graphics (Proc. Siggraph '91)*, 25:61-69, 1991.

[7] R. Wenger. Upper bounds on geometric permutations for convex sets. *Discrete Comput. Geom.*, 5:27–33, 1990.

[8] Yunhong Zhou and Subhash Suri. Shape sensitive geometric permutations. In *Proc. 12th ACM-SIAM Symposium on Discrete Algorithms*, pages 234-243, 2001.

A The Appendix

A.1 Proof of Lemma 2.2

PROOF. We use the line containing o_1, o_2 as our first coordinate axis $\vec{z}$. Thus θ is the angle between ℓ and $\vec{z}$. Let o_1 be the origin, and so $\overrightarrow{o_1o_2} = (L, 0, \cdots, 0)$. Let p_1, p_2 be two arbitrary points where ℓ intersects the balls B_1, B_2. Clearly, $\text{dist}(o_1, p_1) \leq r_1$ and $\text{dist}(o_2, p_2) \leq r_2$. We can write $\overrightarrow{p_1p_2} = \overrightarrow{o_1o_2} + \overrightarrow{o_2p_2} + \overrightarrow{p_1o_1} = (L + x_1, x_2, \cdots, x_d)$, where $x = (x_1, \cdots, x_d) = \overrightarrow{o_2p_2} + \overrightarrow{p_1o_1}$, and so $|x| \leq r_1 + r_2$. Now,

$$\begin{aligned} \cos\theta &= \frac{L(L+x_1)}{L\sqrt{(L+x_1)^2 + x_2^2 + \cdots + x_d^2}} = \frac{L+x_1}{\sqrt{(L+x_1)^2 + x_2^2 + \cdots + x_d^2}} \\ \Rightarrow \quad \sin^2\theta &= \frac{x_2^2 + \cdots + x_d^2}{(L+x_1)^2 + x_2^2 + \cdots + x_d^2} = \frac{y}{(L+x_1)^2 + y}, \end{aligned}$$

where $y = x_2^2 + \cdots + x_d^2$. Denote $f(y) \equiv \frac{y}{c+y}$, with $c = (L + x_1)^2 > 0$, then $f(y)$ is an increasing function of y. Therefore $f(y)$ is maximized when y is maximized. Because $y + x_1^2 = |x|^2 \leq (r_1 + r_2)^2$, thus $y \leq (r_1 + r_2)^2 - x_1^2$. Therefore

$$\begin{aligned} \sin^2\theta &= \frac{y}{(L+x_1)^2 + y} \\ &\leq \frac{(r_1+r_2)^2 - x_1^2}{L^2 + 2Lx_1 + (r_1+r_2)^2} \\ &\leq \frac{(r_1+r_2)^2}{L^2}. \end{aligned}$$

The last inequality follows from the following observations:

$$\begin{aligned} & \frac{(r_1+r_2)^2 - x_1^2}{L^2 + (r_1+r_2)^2 + 2Lx_1} \leq \frac{(r_1+r_2)^2}{L^2} \\ \Leftrightarrow \quad & \left((r_1+r_2)^2 - x_1^2\right) L^2 \leq \left(L^2 + (r_1+r_2)^2 + 2Lx_1\right)(r_1+r_2)^2 \\ \Leftrightarrow \quad & \left(Lx_1 + (r_1+r_2)^2\right)^2 \geq 0. \end{aligned}$$

Thus,

$$\sin^2\theta \leq \frac{(r_1+r_2)^2}{L^2} \Longrightarrow \sin\theta \leq \frac{r_1+r_2}{L}.$$

And it can be verified that equality holds when ℓ is a common tangent of these two balls passing through their geometric center. □

A.2 Proof of Lemma 2.4

PROOF. In order for line ℓ to intersect the unit ball B, it is necessary and sufficient that

$$\min_{t \in \mathcal{R}} \left[(t - o_1)^2 + (at + b - o_2)^2 + \sum_{3 \leq i \leq d} (c_i - o_i)^2 \right] \leq r^2.$$

We note that

$$\begin{aligned} f(t) &= (t-o_1)^2 + (at+b-o_2)^2 \\ &= (a^2+1)t^2 - 2t(o_1 + a(o_2-b)) + o_1^2 + (o_2-b)^2 \\ &= (a^2+1)\left(t - \frac{o_1 + a(o_2-b)}{a^2+1}\right)^2 + \frac{(ao_1 - (o_2-b))^2}{1+a^2}. \end{aligned}$$

Thus, the minimal value of $f(t)$ is

$$\frac{(ao_1 - (o_2-b))^2}{1+a^2},$$

and it is reached when

$$t = t_0 \equiv \frac{o_1 + a(o_2-b)}{a^2+1}.$$

So, in order for ℓ to intersect B, it is necessary and sufficient that $\ell(t_0) \in B$, or equivalently

$$\frac{(ao_1 - (o_2-b))^2}{1+a^2} + \sum_{3 \le i \le d} (c_i - o_i)^2 \le r^2.$$

□

A.3 Proof of Lemma 2.5

Proof.

$$\begin{aligned} \lambda_1(x-a)^2 + \lambda_2(x-b)^2 &= (\lambda_1+\lambda_2)x^2 - 2x(\lambda_1 a + \lambda_2 b) + \lambda_1 a^2 + \lambda_2 b^2 \\ &= (\lambda_1+\lambda_2)\left(x - \frac{\lambda_1 a + \lambda_2 b}{\lambda_1+\lambda_2}\right)^2 + \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(a-b)^2 \\ &\ge \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(a-b)^2. \end{aligned}$$

□

A.4 Proof of Lemma 2.6

Proof. Suppose that $p = (p_1, p_2, \cdots, p_d)$ and $q = (q_1, q_2, \cdots, q_d)$. The geometric center for P and Q is $(r_q p + r_p q)/(r_p + r_q)$. Thus $r_q p_1 + r_p q_1 = 0$, because the first coordinate is zero. Rewrite $p_1 = -r_p\,\epsilon$, $q_1 = r_q\,\epsilon$ for some parameter ϵ. Because the first coordinate axis $\vec{z}$ intersects both P and Q, we have

$$\sum_{2 \le i \le d} p_i^2 \le r_p^2 \qquad \text{and} \qquad \sum_{2 \le i \le d} q_i^2 \le r_q^2. \tag{3}$$

And because it intersects P before Q, we have $-r_p\,\epsilon < r_q\,\epsilon$, or $\epsilon > 0$.

Because line ℓ intersects Q, from Lemma 2.4, $\ell(t_2) \in Q$, where $t_2 = (r_q\,\epsilon + aq_2 - ab)/(a^2+1)$, and

$$\frac{(a\,r_q\,\epsilon + b - q_2)^2}{1+a^2} + \sum_{3 \le i \le d} (c_i - q_i)^2 \le r_q^2. \tag{4}$$

Similarly, because ℓ intersects P, thus $\ell(t_1) \in P$, where $t_1 = (-r_p\ \epsilon + ap_2 - ab)/(a^2+1)$, and

$$\frac{(-a\ r_p\ \epsilon + b - p_2)^2}{1+a^2} + \sum_{3\le i\le d} (c_i - p_i)^2 \le r_p^2. \tag{5}$$

Because ℓ intersect Q first, we have $t_2 < t_1$. We plug the expressions for t_1, t_2 and get

$$\frac{r_q\ \epsilon + aq_2 - ab}{a^2+1} < \frac{-r_p\ \epsilon + ap_2 - ab}{a^2+1} \implies (r_p + r_q)\ \epsilon < a(p_2 - q_2).$$

We multiply Eq. (4) by r_p, multiply Eq. (5) by r_q, and sum the two to get

$$\begin{aligned} &\frac{r_p(ar_q\epsilon + b - q_2)^2 + r_q(-ar_p\epsilon + b - p_2)^2}{1+a^2} + \sum_{3\le i\le d} \left[r_p(c_i - q_i)^2 + r_q(c_i - p_i)^2\right] \\ \le\ & r_p r_q^2 + r_q r_p^2 \ =\ r_p r_q (r_p + r_q). \end{aligned}$$

By Lemma 2.5,

$$r_p\ (c_i - q_i)^2 + r_q\ (c_i - p_i)^2 \ge \frac{r_p\ r_q\ (p_i - q_i)^2}{r_p + r_q}, \quad 3 \le i \le d.$$

Therefore

$$\begin{aligned} &\frac{r_p(ar_q\epsilon + b - q_2)^2 + r_q(-ar_p\epsilon + b - p_2)^2}{1+a^2} + \frac{r_p\ r_q}{r_p + r_q} \sum_{3\le i\le d} (p_i - q_i)^2 \le r_p r_q (r_p + r_q) \\ \Leftrightarrow\ &\frac{r_p(ar_q\epsilon + b - q_2)^2 + r_q(-ar_p\epsilon + b - p_2)^2}{1+a^2} \ \le\ \frac{r_p r_q}{r_p + r_q}\left((r_p + r_q)^2 - \sum_{3\le i\le d} (p_i - q_i)^2\right) \end{aligned} \tag{6}$$

Because P, Q are disjoint, we have

$$(r_p + r_q)^2\epsilon^2 + \sum_{2\le i\le d} (p_i - q_i)^2 \ge (r_p + r_q)^2 \tag{7}$$

$$\Leftrightarrow\quad (r_p + r_q)^2 - \sum_{3\le i\le d} (p_i - q_i)^2 \le (r_p + r_q)^2\epsilon^2 + (p_2 - q_2)^2. \tag{8}$$

By combining (6) and (8) together, we get

$$\begin{aligned} &\frac{r_p(ar_q\epsilon + b - q_2)^2 + r_q(-ar_p\epsilon + b - p_2)^2}{1+a^2} \ \le\ \frac{r_p r_q}{r_p + r_q}\left((r_p + r_q)^2\epsilon^2 + (p_2 - q_2)^2\right) \\ \Leftrightarrow\ &(r_p + r_q)b^2 - 2b(r_q p_2 + r_p q_2) + r_p r_q (r_p + r_q)a^2\epsilon^2 + (r_q p_2^2 + r_p q_2^2) + 2r_p r_q a\epsilon(p_2 - q_2) \\ &\le\ r_p r_q (r_p + r_q)(1+a^2)\epsilon^2 + \frac{r_p r_q}{r_p + r_q}(p_2 - q_2)^2 + \frac{r_p r_q}{r_p + r_q}a^2(p_2 - q_2)^2 \\ \Leftrightarrow\ &(r_p + r_q)b^2 - 2b(r_q\ p_2 + r_p\ q_2) + \frac{(r_q p_2 + r_p q_2)^2}{r_p + r_q} \\ &\le\ r_p r_q (r_p + r_q)\epsilon^2 - 2r_p r_q a\epsilon(p_2 - q_2) + \frac{r_p r_q}{r_p + r_q}a^2(p_2 - q_2)^2 \\ \Leftrightarrow\ &(r_p + r_q)\left(b - \frac{r_q p_2 + r_p q_2}{r_p + r_q}\right)^2 \le \frac{r_p r_q}{r_p + r_q}\left[a(p_2 - q_2) - (r_p + r_q)\epsilon\right]^2 \\ \Leftrightarrow\ &\left|b - \frac{r_q p_2 + r_p q_2}{r_p + r_q}\right| \le \frac{\sqrt{r_p r_q}}{r_p + r_q}\left[a(p_2 - q_2) - (r_p + r_q)\epsilon\right] \end{aligned} \tag{9}$$

From (3), thus

$$\begin{aligned}
& \sum_{2\le i\le d} (r_q p_i + r_p q_i)^2 + r_p r_q \sum_{2\le i\le d} (p_i - q_i)^2 \\
= & \sum_{2\le i\le d} \left[(r_q p_i + r_p q_i)^2 + r_p r_q (p_i - q_i)^2 \right] \\
= & (r_p + r_q) r_q \sum_{2\le i\le d} p_i^2 + (r_p + r_q) r_p \sum_{2\le i\le d} q_i^2 \\
\le & r_p r_q (r_p + r_q)^2
\end{aligned}$$

From (7), thus

$$\sum_{2\le i\le d} (r_q p_i + r_p q_i)^2 \le r_p r_q (r_p + r_q)^2 - r_p r_q \sum_{2\le i\le d} (p_i - q_i)^2 \le r_p r_q (r_p + r_q)^2 \epsilon^2. \tag{10}$$

Eq. (10) implies that

$$\left| \frac{r_q p_2 + r_p q_2}{r_p + r_q} \right| \le \sqrt{r_p\ r_q}\ \epsilon.$$

Combine this with (9), therefore

$$\begin{aligned}
|b| & \le \left| \frac{r_q p_2 + r_p q_2}{r_p + r_q} \right| + \sqrt{r_p\ r_q}\ a\ \frac{(p_2 - q_2)}{r_p + r_q} - \sqrt{r_p\ r_q}\ \epsilon \\
& \le \sqrt{r_p\ r_q}\ a \frac{(p_2 - q_2)}{r_p + r_q} \ \le\ \sqrt{r_p\ r_q}\ a.
\end{aligned}$$

This completes the proof. □

A.5 Proof of Lemma 2.8

Proof. Based on the proof of Lemma 2.7, we get

$$\theta \ \ge\ \pi - \theta' - 2\phi \ \ge\ \frac{\pi}{2} - 2\phi \ =\ \frac{\pi}{2} - O\left(\frac{\gamma^d}{n}\right).$$

□

A.6 Proof of Lemma 2.9

Proof. Because E is the geometric center for $\{B, B_1\}$, thus o, E, o_1 are collinear. Because $\mathrm{dist}(B, B_1) = 0$, thus $\mathrm{dist}(o, E) = r$, $\mathrm{dist}(E, o_1) = r_1$, and $\mathrm{dist}(o, o_1) = r + r_1$. Similarly o, F, o_2 are collinear, $\mathrm{dist}(o, F) = r$, $\mathrm{dist}(F, o_2) = r_2$, and $\mathrm{dist}(o, o_2) = r + r_2$. See Figure 2 for an illustration. Because $\mathrm{dist}(B_1, B_2) \ge 0$, thus $\mathrm{dist}(o_1, o_2) \ge r_1 + r_2$. It is obvious that o, E, F, o_1, o_2 are coplanar. Let θ denote the angle between oo_1 and oo_2, then

$$\cos\theta = \frac{\mathrm{dist}(o, o_1)^2 + \mathrm{dist}(o, o_2)^2 - \mathrm{dist}(o_1, o_2)^2}{2\mathrm{dist}(o, o_1)\mathrm{dist}(o, o_2)} \le \frac{(r + r_1)^2 + (r + r_2)^2 - (r_1 + r_2)^2}{2(r + r_1)(r + r_2)}.$$

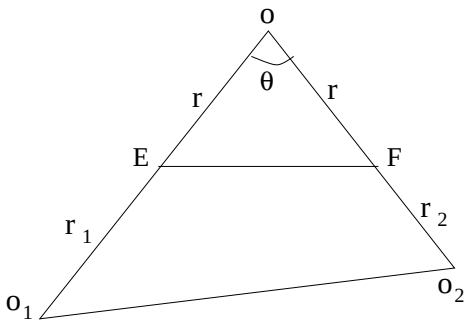

Figure 2: Illustration of the distance between two geometric centers.

Thus

$$\begin{aligned}
\text{dist}(E,F)^2 &= \text{dist}(o,E)^2 + \text{dist}(o,F)^2 - 2\text{dist}(o,E)\text{dist}(o,F)\cos\theta \\
&= 2r^2(1-\cos\theta) \\
&\geq 2r^2\left(1 - \frac{(r+r_1)^2 + (r+r_2)^2 - (r_1+r_2)^2}{2(r+r_1)(r+r_2)}\right) \\
&= 2r^2\left(\frac{(r_1+r_2)^2 - (r_1-r_2)^2}{2(r+r_1)(r+r_2)}\right) \\
&= \frac{4r^2 r_1 r_2}{(r+r_1)(r+r_2)} = \frac{2rr_1}{r+r_1}\,\frac{2rr_2}{r+r_2} \geq 1
\end{aligned}$$

Therefore $\text{dist}(E,F) \geq 1$ and the proof is complete. □

A.7 Proof of Lemma 2.10

Proof. We prove the unique pairing property of switched pairs by contradiction. Suppose that ball B forms switched pairs with at least two other balls. Specifically, assume that both (B, B_1) and (B, B_2) are switched pairs, where B, B_1, B_2 are distinct balls in $\mathcal{S}$, with centers o, o_1, o_2 and radius r, r_1, r_2 respectively.

From Lemma 2.7, $\text{dist}(B, B_1) = O(\gamma^{2d+1}/n^2)$, and $\text{dist}(B, B_2) = O(\gamma^{2d+1}/n^2)$. Because $n = \omega(\gamma^{d+1/2})$, both $\text{dist}(B, B_1), \text{dist}(B, B_2)$ go to zero when n is large. In the limit $\text{dist}(o, o_1) = r + r_1$, and $\text{dist}(o, o_2) = r + r_2$. Let $\vec{z}$ denote the line connecting the longest center pair of $\mathcal{S}$. From Lemma 2.8, both $\overrightarrow{oo_1}$ and $\overrightarrow{oo_2}$ are almost perpendicular to $\vec{z}$. Let E be the geometric center for $\{B, B_1\}$. Similarly F is the geometric center for $\{B, B_2\}$. Suppose that π is the plane passing through o and perpendicular to $\vec{z}$, then both o_1, o_2 are very close to π. EF is almost perpendicular to $\vec{z}$ and contained in π. Consider the limiting case of $n = \infty$, which is described in Lemma 2.9. From Lemma 2.9, thus $\text{dist}(E, F) \geq 1$ under the limit circumstance. From Lemma 2.6, for any line transversal ℓ, it should pass close to E, the geometric center of $\{B, B_1\}$, as well as F, the geometric center of $\{B, B_2\}$. So that ℓ should be almost parallel to EF, or equivalently, ℓ should be almost perpendicular to $\vec{z}$. This is a contradiction, as the angle between ℓ and $\vec{z}$ is bounded by $O(\gamma^d/n)$, from Lemma 2.3. □

A.8 Proof of Lemma 4.1

PROOF. Let us consider the limit case first, i.e., the case where $n = \infty$. In the limit, each line transversal coincides with a fixed directed line z' which is parallel to the first coordinate axis $\vec{z}$. And in the limit, for each switched pair, their geometric center should lie in z'. These two balls are tangent to each other, with the line connecting their centers perpendicular to $\vec{z}$. For simplicity of notation, we can assume that z' coincides with $\vec{z}$.

Now consider these two switched pairs at the limit case. We assume that $o_1 = (0, r_1\, x_2, \cdots, r_1\, x_d)$ and $o_1' = (0, -r_1'\, x_2, \cdots, -r_1'\, x_d)$. Here the geometric center for $\{B_1, B_1'\}$ is the origin $(0, \cdots, 0)$ and $x_2^2 + \cdots + x_d^2 = 1$. As for the second switched pair, suppose that its geometric center has height L. Then we can assume that $o_2 = (L, r_2\ y_2, \cdots, r_2\ y_d)$ and $o_2' = (L, -r_2'\ y_2, \cdots, -r_2'\ y_d)$. At the same time, $y_2^2 + \cdots + y_d^2 = 1$. Because all balls are pairwise disjoint, thus we get the following two inequalities:

$$L^2 + (r_1\ x_2 - r_2\ y_2)^2 + \cdots + (r_1\ x_d - r_2\ y_d)^2 \ \geq\ (r_2 + r_1)^2\ , \tag{11}$$

$$L^2 + (r_1'\ x_2 + r_2\ y_2)^2 + \cdots + (r_1'\ x_d + r_2\ y_d)^2 \ \geq\ (r_2 - r_1')^2\ . \tag{12}$$

Inequality (11) times r_1' plus (12) times r_1, then

$$\begin{aligned}
&(r_1 + r_1')L^2 + \left(r_1^2 r_1' + (r_1')^2 r_1\right)(x_2^2 + \cdots + x_d^2) + r_2^2(r_1 + r_1')(y_2^2 + \cdots + y_d^2) \\
&\geq r_1'(r_2 + r_1)^2 + r_1(r_2 + r_1')^2 \\
\Leftrightarrow\ &(r_1 + r_1')L^2 + \left(r_1^2 r_1' + (r_1')^2 r_1\right) + r_2^2(r_1 + r_1')\ \geq\ r_1'(r_2 + r_1)^2 + r_1(r_2 + r_1')^2 \\
\Rightarrow\ &L^2 \geq \frac{4r_2\ r_1\ r_1'}{r_1 + r_1'}\ .
\end{aligned}$$

As this is the limit case, so that for real circumstance, the following holds

$$L \geq \sqrt{\frac{4r_2\ r_1\ r_1'}{r_1 + r_1'}} - \epsilon(n), \text{ where } \lim_{n\to\infty} \epsilon(n) = 0.$$

□

A.9 Proof of Lemma 4.2

PROOF. By Lemma 3.3, there exist two line transversals ℓ, ℓ' and they cut both switched pairs in different orders. Let us choose ℓ as the first coordinate axis, and choose the projection of ℓ' into the hyperplane perpendicular to ℓ as the second coordinate axis. Thus ℓ, ℓ' can be expressed analytically as the following:

$$\begin{aligned}
\ell\ &:\ \{\ell(t) = (t, 0, \cdots, 0) \mid t \in \mathcal{R}\}, \\
\ell'\ &:\ \{\ell'(t) = (t, at + b, c_3, \cdots, c_d) \mid t \in \mathcal{R}\},
\end{aligned}$$

where $a, b, c_3, \cdots, c_d$ are all constants.

Let g_i denote the geometric center for switched pair $\{B_i, B_i'\}$, for $i = 1, 2$. Suppose that the first coordinate of g_1 is x_{g_1}, and the first coordinate of g_2 is x_{g_2}. In addition, we can assume that $x_{g_1} < x_{g_2}$—that is, line ℓ meets switched pair $\{B_1, B_1'\}$ before $\{B_2, B_2'\}$. Because ℓ, ℓ' meet both switched pairs in different order, thus from Lemma 2.6, we have

$$|ax_{g_1} + b| \leq \sqrt{r_1\ r_1'}\ a \quad \text{and} \quad |ax_{g_2} + b| \leq \sqrt{r_2\ r_2'}\ a.$$

Combine these two inequalities together, thus

$$a|x_{g_2} - x_{g_1}| \leq \left(\sqrt{r_1\ r_1'} + \sqrt{r_2\ r_2'}\right)\ a \quad \Rightarrow \quad x_{g_2} - x_{g_1} \leq \sqrt{r_1\ r_1'} + \sqrt{r_2\ r_2'}.$$

The distance between g_1, g_2 is dominated by the difference between their first coordinates. Thus

$$\text{dist}(g_1, g_2) \leq \sqrt{r_1\ r_1'} + \sqrt{r_2\ r_2'} + \epsilon(n), \text{ where } \lim_{n\to\infty} \epsilon(n) = 0.$$

□

A.10 Proof of Lemma 4.4

Proof. Recall that the Hamming distance is also the number of different coordinates between x and y. Let us prove this property now. Suppose that $\text{dist}_H(x, y) = \delta > 2(1 + \lfloor \log \gamma \rfloor)$ for some numbers $x, y \in H^m$. Let ℓ, ℓ' denote two line transversals corresponding to x, y respectively. Then there exist δ switched pairs and ℓ, ℓ' cut them in different orders. We relabel these δ switched pairs in sorted height and still use $SP_1, SP_2, \cdots, SP_\delta$ to denote them. We transform the coordinate system such that ℓ is the first coordinate axis, and ℓ' is the following:

$$\ell' \ : \ \{\ell'(t) = (t, at + b, c_3, \cdots, c_d) \mid t \in \mathcal{R}\}.$$

Let g_i denote the geometric center for SP_i and x_{g_i} denote the first coordinate of g_i. From Lemma 2.6, we have

$$|ax_{g_i} + b| \leq \sqrt{r_i\ r_i'}\ a, \qquad \text{for all } 1 \leq i \leq \delta.$$

Let $b' = b/a$, then $|x_{g_i} + b'| \leq \sqrt{r_i\ r_i'}$ for all i. For the set $\{x_{g_i} + b' \mid 1 \leq i \leq \delta\}$, there are at least $\lceil \delta/2 \rceil$ numbers to be either all nonnegative or all nonpositive. Without loss of generality, we can assume that there are at least $\lceil \delta/2 \rceil$ non-negative numbers. For convenience, relabel these switched pairs to be $SP_1, SP_2, \cdots, SP_{\delta'}$. Here $\delta' \geq \lceil \delta/2 \rceil \geq 2 + \lfloor \log \gamma \rfloor$. And $0 \leq x_{g_i} + b' \leq \sqrt{r_i\ r_i'}$, for $1 \leq i \leq \delta'$.

Lemma 4.1 gives us a lower bound on $x_{g_{i+1}} - x_{g_i}$. Combining it with the condition that $0 \leq x_{g_i} + b'$ leads to the following:

$$\sqrt{\frac{4r_i\ r_{i+1}\ r_{i+1}'}{r_{i+1} + r_{i+1}'}} \leq x_{g_{i+1}} - x_{g_i} + x_{g_i} + b' = x_{g_{i+1}} + b' \leq \sqrt{r_{i+1}\ r_{i+1}'}.$$

The above inequality implies that $4r_i \leq r_{i+1} + r_{i+1}' \leq 2r_{i+1}$, or $2r_i \leq r_{i+1}$. So that

$$r_{\delta'} \ \geq \ r_1 2^{\delta'-1} \ \geq \ 2^{1+\lfloor \log \gamma \rfloor} \ > \ \gamma.$$

It contradicts with the condition that $\mathcal{S}$ has radial ratio γ. So that the claim is proved. □