Geometric Permutations of Balls with Bounded Size Disparity

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Abstract

We study combinatorial bounds for geometric permutations of balls with bounded size disparity in d-space. Our main contribution is the following theorem: given a set $\mathcal{S}$ of n disjoint balls in $\mathcal{R}^d$, if n is sufficiently large and the radius ratio between the largest and smallest balls of $\mathcal{S}$ is γ, then the maximum number of geometric permutations of $\mathcal{S}$ is $O(\gamma^{\log \gamma})$. When $d = 2$, we are able to prove the tight bound of 2 on the number of geometric permutations for $\mathcal{S}$, which is the best possible bound because it holds even when $\gamma = 1$. Our theorem shows how the number of permutations varies as a function of the size disparity among balls, thus gracefully bridging the gap between two extreme bounds that were known before: the $O(1)$ bound for congruent balls, and the $\Theta(n^{d-1})$ bound for arbitrary balls.

1 Introduction

A line ℓ is called a *line transversal* for a set $\mathcal{S}$ of bodies in $\mathcal{R}^d$ if it intersects every member of $\mathcal{S}$. If the objects in $\mathcal{S}$ are pairwise disjoint convex bodies, then a line transversal of $\mathcal{S}$ induces two linear orderings, which are the orders in which ℓ meets the members of $\mathcal{S}$. The two orders induced by a line transversal are essentially the same (one is reverse of the other), and so they are together called a single *geometric permutation*. Let $g_d(n)$ denote the maximum number of geometric permutations, where the maximum is taken over all such families $\mathcal{S}$ of size n in $\mathcal{R}^d$. Then, it is known that $g_2(n) = 2n - 2$ [2], and that $g_d(n)$ is in $\Omega(n^{d-1})$ [4] and $O(n^{2d-2})$ [7].

Thus, for dimensions $d \geq 3$, there is a substantial gap between the known upper and lower bounds on geometric permutations. This gap motivated Smorodinsky, Mitchell, and Sharir [5] to consider a natural special case of convex bodies: *balls*. They proved that the maximum number of geometric permutations achieved by n pairwise disjoint balls in $\mathcal{R}^d$ is $O(n^{d-1})$. Moreover, this bound is tight as they were able to show a matching lower bound construction.

An interesting special case of the problem is when all the balls in $\mathcal{S}$ are congruent (say, unit radius balls). It was conjectured that for unit radius balls, the number of geometric permutations drops precipitously to $O(1)$, with the constant depending on d. The *planar* case of this conjecture was proved by Katchalski and Asinowski [1] and also by Smorodinsky et al. [5]. They showed that when $|\mathcal{S}|$ is sufficiently large, the maximum number of geometric permutations admitted by n unit disks is *two*.

Recently, we proved the unit radius ball conjecture for arbitrary dimensions, and showed that for n congruent balls in $\mathcal{R}^d$, the maximum number of geometric permutations is *four* [8], where n, the size of the set $|\mathcal{S}|$, depends on the dimension constant d. Note that unlike the conjecture made by Smorodinsky, Mitchell, and Sharir [5], the bound on the geometric permutations is *independent* of d.

The sharp difference between the bounds for congruent and arbitrary balls is quite intriguing. The bound jumps abruptly from four to $\Theta(n^{d-1})$ when the congruency constraint is removed. For instance, even in the case of planar disks, the number of permutations is at most 2 if the disks are congruent, but it can be $\Theta(n)$ otherwise. A closer examination of the problem reveals that constructions that achieve $\Omega(n)$ distinct permutations for disks require *exponentially varying* radii, and the same phenomenon occurs for higher dimensions as well. Thus, both from a practical as well as a mathematical perspective, it is an open question as to how the number of permutations grows as a function of the varying radii. In particular, is the number of permutations a constant if the ratio between the largest and smallest balls is bounded by a constant?

Our Contributions

In this paper, we investigate the geometric permutation problem for disjoint balls in $\mathcal{R}^d$ with bounded size disparity. Our bound depends on the radius ratio between the largest and the smallest balls. Specifically, assume that $\mathcal{S} = \{B_1, \cdots, B_n\}$ is a set of disjoint balls in $\mathcal{R}^d$, where $B_i \equiv B(o_i, r_i)$, meaning that B_i has center o_i and radius r_i. We define $\gamma(\mathcal{S})$ as the *radial ratio* of $\mathcal{S}$:

$$\gamma(\mathcal{S}) = \max_{i,j} \frac{r_i}{r_j}.$$

Our main result is the following: Let $n = |\mathcal{S}|$ and $\gamma = \gamma(\mathcal{S})$. If n is sufficiently large[1], then the maximum number of geometric permutations of $\mathcal{S}$ is $O(\gamma^{\log \gamma})$. When $d = 2$, we are able to prove the tight bound of 2 on the number of geometric permutations for $\mathcal{S}$, which is the best possible bound because it holds even when $\gamma = 1$.

Our bounds provide a more refined analysis of the geometric permutation problem, showing how the number of permutations varies as a function of the size disparity among balls, and gracefully bridging the gap between two extreme bounds that were known before: the $O(1)$ bound for congruent balls, and the $\Theta(n^{d-1})$ bound for arbitrary balls.

A simple, but quite practical, corollary of our result is the following. Let $\mathcal{S}$ be a set n pairwise disjoint convex bodies in d-space such that their *smallest enclosing balls* are also pairwise disjoint. If the radial ratio of the enclosing balls is bounded by γ, then the number of geometric permutations for $\mathcal{S}$ is $O(\gamma^{\log \gamma})$. In other words, even arbitrary convex bodies admit only a constant number of permutations *if* they are relatively well-separated and have bounded size disparity. By contrast, without these assumptions, the best upper bound known for permutations of convex objects is $O(n^{2d-2})$, and there is a lower bound of $\Omega(n^{d-1})$.

While in combinatorial geometry line transversals and geometric permutations have been studied primarily for their mathematical appeal, they also find practical applications in computing visibility information in 3-dimensional computer games and architectural walkthroughs. These applications decompose the space into boxes, and then try to compute visibility between various boxes. The openings through which light can pass, such as windows, doors and the tops of stairwells in a building model, are modeled by convex *portals* on the faces of these boxes; all other objects are considered opaque. The visibility among boxes is computed from the visibility between these portals. See Teller and Sequin [6] for details. Similar ideas are used in various video games as well.

2 Properties of Line Transversals

Let $\mathcal{S} = \{B_1, \cdots, B_n\}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ and $B_i = B(o_i, r_i)$ for $1 \le i \le n$. With proper scaling, we can assume that $1 \le r_i \le \gamma$, for $1 \le i \le n$. Our analysis relies on the critical assumption that n is large enough, and that γ is fixed. We will slightly abuse the notation and write *diameter* of $\mathcal{S}$ to mean the diameter of the set of centers of all the balls in $\mathcal{S}$. That is, $D(\mathcal{S}) = \max_{ij} \operatorname{dist}(o_i, o_j)$. (Recall that $f(n) = \omega(g(n)) \Longleftrightarrow \lim_{n\to\infty} f(n)/g(n) = \infty$.)

Lemma 2.1 *Let $\mathcal{S}$ be a set of n disjoint balls in $\mathcal{R}^d$ with radial ratio γ. If $\mathcal{S}$ admits at least one geometric permutation and $n = \omega(\gamma^d)$, then the diameter of $\mathcal{S}$ is $\Omega(n/\gamma^{d-1})$.*

Proof. Let ℓ be a line transversal of $\mathcal{S}$. The projection of $\mathcal{S}$ onto ℓ has length at most $2\gamma + D$, here D is the diameter of $\mathcal{S}$. Thus all members of $\mathcal{S}$ are contained within a radius 2γ cylinder of height $2\gamma + D$. If V_d denotes the volume of a unit ball in $\mathcal{R}^d$, then the volume of this cylinder is $(2\gamma + D)V_{d-1}(2\gamma)^{d-1}$. Since n disjoint balls are contained in this cylinder and d is a constant, we get

$$(2\gamma + D)V_{d-1}(2\gamma)^{d-1} \ge nV_d,$$

which implies that

$$D \ge \frac{V_d}{2^{d-1}V_{d-1}} \frac{n}{\gamma^{d-1}} - 2\gamma = \Omega\left(\frac{n-\gamma^d}{\gamma^{d-1}}\right) = \Omega\left(\frac{n}{\gamma^{d-1}}\right).$$

The last inequality uses the fact that $n = \omega(\gamma^d)$. $\square$

Lemma 2.2 *Let $B_1 = B(o_1, r_1)$ and $B_2 = B(o_2, r_2)$ be two disjoint balls in $\mathcal{R}^d$, and let ℓ be a line that intersects them both. Let $L = \operatorname{dist}(o_1, o_2)$ and let θ denote the angle between ℓ and the line containing o_1, o_2. Then, we have $\sin\theta \le \frac{r_1+r_2}{L}$, and θ achieves its maximum value θ_0 when ℓ is a common tangent of B_1, B_2 passing through their geometric center.*

Lemma 2.3 *Suppose $\mathcal{S}$ is a set of n disjoint balls in $\mathcal{R}^d$ with radial ratio γ. $\mathcal{S}$ accepts at least one geometric permutation with $n = \omega(\gamma^d)$. Let z-axis be the line connecting two centers with the longest distance. Then the angle between any line transversal of $\mathcal{S}$ and z-axis is bounded by $O(\gamma^d/n)$; consequently, the angle between any two line transversals of $\mathcal{S}$ is also $O(\gamma^d/n)$.*

2.1 Geometric Center

Given two disjoint balls $B(o_1, r_1)$ and $B(o_2, r_2)$, their geometric center is defined as $(r_2 o_1 + r_1 o_2)/(r_1 + r_2)$. In the plane, the geometric center of two disjoint disks is the intersection point of their common inner tangents. The geometric center is collinear with the centers of these two balls, and lies outside of both balls.

In order to bound geometric permutations quantitatively, we need a technical lemma to estimate the *closeness* of line transversals to geometric centers.

[1] See Theorem 3.6.

Lemma 2.4 is such a key technical lemma, and it establishes an important algebraic condition for the line transversal of two balls. Intuitively, the lemma shows that if two line transversals intersect two balls in different orders, then these two lines must pass *very close* to the geometric center of these two balls; the closeness depends on the radii of the balls.

Lemma 2.4 *Let $P = B(p, r_p)$ and $Q = B(q, r_q)$ be two disjoint balls in $\mathcal{R}^d$ with centers p, q and radii r_p, r_q respectively. Assume that their geometric center has its first coordinate equal to zero, and that the first coordinate axis $\vec{z}$ intersects both P and Q, with P first. If another oriented line ℓ with parametric form $\{\ell(t) = (t, at+b, c_3, \cdots, c_d) \mid t \in \mathcal{R}\}$ intersects both P and Q, with Q first, then we must have $|b| \leq \sqrt{r_p r_q}\, a$. Here $a > 0$ is the slope of line ℓ, and $c_3, \cdots, c_d$ are all constants.*

The proof of Lemma 2.4 rests on the following two simpler lemmas. For a complete proof, please refer to the (electronic) full version of the paper.

Lemma 2.5 *Let $B(o, r)$ be a ball in $\mathcal{R}^d$ with center $o = (o_1, o_2, \cdots, o_d)$ and radius r. Let $\ell = \{\ell(t) \mid t \in \mathcal{R}\}$ be the parametric equation of a line with $\ell(t) = (t, at + b, c_3, \cdots, c_d)$. If ℓ has a nonempty intersection with B, then $\ell(t_0) \in B$, where $t_0 = \frac{o_1 + a(o_2 - b)}{a^2+1}$, or equivalently*

$$\frac{(ao_1 - (o_2 - b))^2}{1 + a^2} + \sum_{3 \leq i \leq d} (c_i - o_i)^2 \leq r^2 .$$

Lemma 2.6 *Let $\lambda_1, \lambda_2 > 0$ and $a, b \in R$. Then the following inequality holds:*

$$\lambda_1(x-a)^2 + \lambda_2(x-b)^2 \geq \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(a-b)^2, \quad \forall\, x \in R.$$

2.2 Switched Pairs

We say that a pair of balls B_1, B_2 forms a *switched pair* in $\mathcal{S}$ if there exist two line transversals such that they meet B_1, B_2 in different orders. Switched pairs are central to geometric permutations, and our next few lemmas establish key properties that allow us to bound their number. The first property shows that two balls making a switched pair are very close.

Lemma 2.7 *The distance between two balls making a switched pair is $O(\gamma^{2d+1}/n^2)$.*

Lemma 2.8 *Let (B_1, B_2) be a switched pair in $\mathcal{S}$, and let o_1, o_2 be the centers of these balls. If ℓ is a line transversal of $\mathcal{S}$, then the line ℓ is nearly perpendicular to the line passing through o_1, o_2. More specifically, the angle between ℓ and $\overrightarrow{o_1 o_2}$ is $\pi/2 - O(\gamma^d/n)$.*

Before we introduce the unique pairing property for general balls, we need one more technical lemma.

Lemma 2.9 *Let $B = B(o, r), B_1 = B(o_1, r_1), B_2 = B(o_2, r_2)$ be three balls in $\mathcal{R}^d$ with $\operatorname{dist}(B, B_1) = 0, \operatorname{dist}(B, B_2) = 0, \operatorname{dist}(B_1, B_2) \geq 0$, where r, r_1, r_2 are all positive numbers greater than or equal to 1. If E is the geometric center of $\{B, B_1\}$, and F is the geometric center of $\{B, B_2\}$, then $\operatorname{dist}(E, F) \geq 1$.*

The following lemma describes one of the most important properties of switched pairs: *unique pairing property*.

Lemma 2.10 *Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. $\mathcal{S}$ has at least one line transversal with $n = \omega(\gamma^{d+1/2})$. Then, a ball of $\mathcal{S}$ can appear in at most one switched pair.*

The unique pairing property of switched pairs easily implies the fact that if there are m switched pairs, then there can be at most 2^m permutations—each permutation is defined by one of the two orders of some switched pairs.

Lemma 2.11 *Given a set $\mathcal{S}$ of n disjoint convex objects in $\mathcal{R}^d$. If the unique pairing property holds for switched pairs, then two members making a switched pair should always appear consecutively in all line transversals of $\mathcal{S}$.*

The following important fact is easily obtained by combining Lemma 2.10 and Lemma 2.11.

Lemma 2.12 *If $\mathcal{S}$ contains m switched pairs, then $\mathcal{S}$ admits at most 2^m geometric permutations.*

3 Bounds on Geometric Permutations

In the planar case, we are able to establish the tight bound of two permutations.

Theorem 3.1 *Let $\mathcal{S}$ be a set of n disjoint balls in $\mathcal{R}^2$ with radial ratio $\gamma = \gamma(\mathcal{S})$. If $n = \omega(\gamma^{5/2})$, then there are at most two geometric permutations on $\mathcal{S}$.*

When considering geometric permutations in higher dimensions, we transform the coordinate system so that z-axis is oriented along the line connecting two centers with the longest distance in $\mathcal{S}$. For each switched pair, we project its geometric center onto the z-axis and call the projected coordinate the *height* of this switched pair. Thus, all the switched pairs of $\mathcal{S}$ can be linearly ordered using their heights. The key components of our proof are lower and upper bounds on the height distance between geometric centers of two switched pairs. The next two lemmas establish them.

Lemma 3.2 *Let $\{B_1, B_1'\}$ and $\{B_2, B_2'\}$ be two switched pairs of $\mathcal{S}$ with centers o_1, o_1', o_2, o_2' and radii r_1, r_1', r_2, r_2' respectively. The height distance between the geometric centers of these two switched pairs is at least $\sqrt{\frac{4r_2\ r_1\ r_1'}{r_1+r_1'}} - \epsilon(n)$, where $\lim_{n\to\infty}\epsilon(n) = 0$.*

Lemma 3.3 *Let $\{B_1, B_1'\}$ and $\{B_2, B_2'\}$ be two switched pairs of $\mathcal{S}$ with centers o_1, o_1', o_2, o_2' and radii r_1, r_1', r_2, r_2' respectively. The height distance between the geometric centers of these two switched pairs is at most $\sqrt{r_1\ r_1'} + \sqrt{r_2\ r_2'} + \epsilon(n)$, where $\lim_{n\to\infty}\epsilon(n) = 0$.*

Our first theorem gives a weaker, but easier to prove upper bound.

Theorem 3.4 *Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. If $d \geq 3$ is a constant and $n = \omega(\gamma^{d+1/2})$, then the total number of switched pairs is bounded by $1 + \lfloor\sqrt{2}\gamma\rfloor$. Consequently, $\mathcal{S}$ admits at most $2^{1+\lfloor\sqrt{2}\gamma\rfloor}$ geometric permutations.*

The above theorem gives an upper bound of 4 for unit balls when $\gamma = 1$, thus generalizing our result for congruent balls [8]. While this bound is fairly tight when γ is small, its exponential dependence on γ makes it less attractive as γ gets larger. In the following, we improve this bound substantially using Hamming distance to argue about distinct geometric permutations.

Permutation Bounds Using Hamming Distances

Let $H^d = \{(x_1, \cdots, x_d) \mid x_i \in \{0,1\}, 1 \leq i \leq d\}$ denote the d-dimensional Hamming space. For any $x, y \in H^d$, $\mathrm{dist}_H(x, y)$ is the Hamming distance between x and y. Recall that the Hamming distance is the number of different coordinates between x and y.

Lemma 3.5 *Let $x, y \in H^m$ denote the representations of two geometric permutations of $\mathcal{S}$. Then* $\mathrm{dist}_H(x, y) \leq 2(1 + \lfloor\log\gamma\rfloor)$.

Theorem 3.6 *Let $\mathcal{S}$ be a set of n pairwise disjoint balls in $\mathcal{R}^d$ with radial ratio γ. If $d \geq 3$ is a constant and $n = \omega(\gamma^{d+1/2})$, then $\mathcal{S}$ admits $O\left(\gamma^{\log\gamma}\right)$ geometric permutations.*

4 Conclusion

We considered the problem of geometric permutations for balls in d dimensions with bounded size disparity, and offered a substantially refined and shape-sensitive analysis. The previous results for this problem had shown worst-case optimal bounds for two extreme cases of the input variety: for congruent balls, the number of distinct permutations is $O(1)$, while for arbitrary balls, the number of permutations is $\Theta(n^{d-1})$. We considered the natural question: how does the combinatorial complexity of geometric permutations increase as we relax the congruency constraint. Our best upper bound is $O(\gamma^{\log\gamma})$, where γ is radius ratio between the largest and smallest balls of $\mathcal{S}$. In particular, for constant γ, the number of permutations is also a constant.

In the planar case, we were able to prove the tight bound of 2. Thus, for any set of n disks with radius ratio γ and $n = \omega(\gamma^{5/2})$, the number of permutations is at most 2.

The idea of incorporating shape-sensitivity in the analysis of geometric algorithms seems to be a powerful and fruitful one. There appear to be many cases where the bounds obtained through the traditional analysis are quite pessimistic—the constructions creating lower bounds are highly pathological and involve exponential shape or size disparity, which may be extremely unlikely in practice. A more refined analysis, like the one offered in this paper, can provide a more useful insight into the problem complexity.

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