Tight Degree Bounds for Pseudo-triangulations of Points

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Abstract

We show that every point set in general position has a minimum pseudo-triangulation whose maximum vertex degree is five. This bound is tight.

Furthermore we illustrate that every point set in general position also has a minimum pseudo-triangulation whose maximum face degree is four (i.e. each face of this pseudo-triangulation has at most four vertices).

1 Introduction

A *pseudo-triangle* is a planar polygon that has exactly three convex vertices, called *corners,* with internal angles less than π. A *pseudo-triangulation* for a set P of n points in the plane is a partition of the convex hull of P into pseudo-triangles whose vertex set is exactly P.

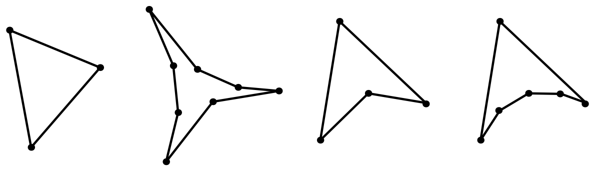

Figure 1: Pseudo-triangles.

Pseudo-triangulations, also called geodesic triangulations, have been studied for convex sets and for simple polygons in the plane because of their applications to visibility [7, 9] and ray shooting [2, 3]. Recently, they have been used in a number of kinetic data structures (KDSs) for collision detection among moving polygons in the plane [1, 5, 6] because they can easily be maintained by edge swaps as points move. Furthermore, one-degree-of-freedom mechanisms induced by pseudo-triangulations can provide efficient algorithms for robot arm motion planning problems [10].

Of particular interest are *minimum* pseudo-triangulations, since they have the least number of faces and edges among all pseudo-triangulations of the same point set. To be specific, a pseudo-triangulation is minimum if it uses $n-2$ pseudo-triangles. This definition of Streinu [10] was adapted from Pocchiola and Vegter [8]. Equivalently a pseudo-triangulation is minimum if and only if every vertex v has one incident region whose angle at v is greater than π. Since every point set in general position has a minimum pseudo-triangulation, we will restrict ourselves to the study of minimum pseudo-triangulations (which we will simply refer to as pseudo-triangulations for the remainder of this paper).

Even though the average degree of a standard triangulation is $O(1)$ there are sets of n points in the plane for which each possible triangulation contains a vertex of degree $n-1$. On the other hand, every point set in the plane has a pseudo-triangulation of constant maximum vertex degree. Kettner et al. [4] showed that there are sets of points for which any pseudo-triangulation has a vertex of degree at least five. They also provided an upper bound construction that results in a pseudo-triangulation of vertex degree at most six. We close this gap and give a tight upper bound construction that yields a pseudo-triangulation of maximum vertex degree five. Note that in fact our upper bound construction does not use the full flexibility of generic pseudo-triangles but instead produces only pseudo-triangles with two straight sides (see Fig. 1(iv)).

A natural question to ask is whether for every set of points there also exists a pseudo-triangulation with a bounded *face degree*, i.e. a pseudo-triangulation with a bounded number of vertices on each face. We describe a construction that yields a pseudo-triangulation in which each face contains either three or four vertices (see Fig. 1(i) and 1(iii)).

2 Pseudo-triangulation of bounded vertex degree

We describe a recursive construction of a pseudo-triangulation for a set of n points in general position. Starting with the convex hull of the points we choose, at each step of the recursion, one of three possible operations each of which reduces the size of the problem in a different manner.

More specifically, for a given convex polygon P with k vertices and i internal points let $s(P) = k+i$ denote the *size* of the problem associated with pseudo-triangulating P. The first operation uses two pseudo-triangles to *partition* P into two vertex-disjoint convex polygons P_1 and P_2 (see Fig. 2(i)) where

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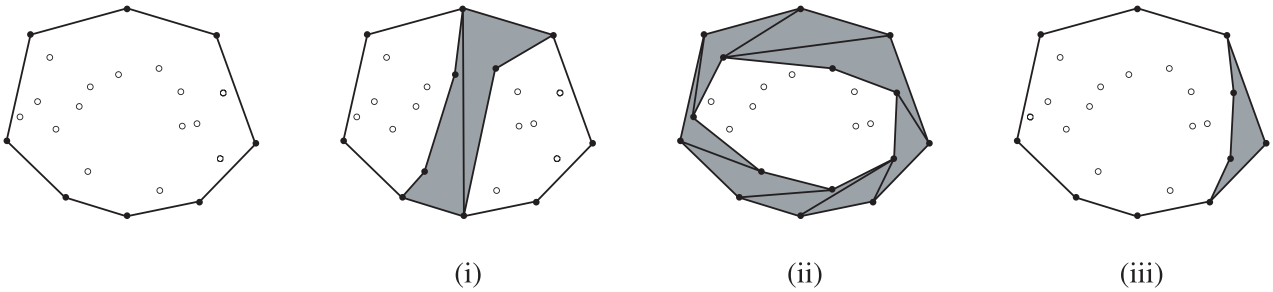

Figure 2: The three operations: (i) partition, (ii) shell and (iii) prune. Note that the shaded pseudo-triangles do not contain any points in their interior.

$\max\{s(P_1), s(P_2)\} \leq s(P) - 1$. The second operation creates a *shell* inside P, i.e. it forms the convex hull P' of the internal points of P and pseudo-triangulates the area between P and P' (see Fig. 2(ii)). Here we have $s(P') = i \leq s(P) - 3$. The third operation uses a geodesic to *prune* a vertex from P and form a convex polygon P' with $s(P') = s(P) - 1$ (see Fig. 2(iii)).

Now consider a polygon P that describes a subproblem. Since no vertex in the final pseudo-triangulation can have degree greater than five the incoming edges incident with vertices on the boundary of P constrain the choice of pseudo-triangulation edges for P. We define the *load* of a vertex v denoted by $l(v)$ to be the degree of v minus two (for example a vertex of degree four has a load of two and a vertex of degree two has no load at all). The load of a polygon P denoted by $l(P)$ is accordingly defined as the sum of the loads of the vertices on its boundary.

We show how to implement the reduction step so as to maintain the following invariants for all polygons P that arise as recursive subproblems:

Invariant 1 $l(P) \leq 6$.

Invariant 2 *No vertex of P has a load greater than 3 and at most one vertex has load 3.*

Both invariants are trivially true for the convex hull of the set of points.

Let us now assume that we are given a convex polygon P that satisfies both invariants. The appropriate operation to choose depends on both the number of vertices of P and the distribution of its load. We distinguish three different cases:

1. P is a triangle $\Rightarrow$ **shell**.
2. P is not a triangle and has vertices of load two or three (i.e. vertices of degree four or five) $\Rightarrow$ **prune**.
3. P is not a triangle and has only vertices of load at most one (i.e. only vertices of degree at most three) $\Rightarrow$ **partition**.

The following sections explain all three cases and operations in detail and illustrate how they maintain the invariants.

2.1 Partition

Assume we are given a convex polygon P that satisfies both invariants, is not a triangle (i.e. P has at least four vertices on its boundary) and has only vertices of load at most one (i.e. only vertices of degree at most three). This implies that P contains at most six vertices of degree three on its boundary.

We choose two of the vertices of P, v_1 and v_2, that are not neighbours, such that they divide the boundary of P into two polygonal chains that each contain at most two vertices of degree three (apart from v_1 and v_2). Note that if P contains exactly six vertices of degree three then v_1 and v_2 have to be of degree three, otherwise either one (if P contains five vertices of degree three) or both of them can be of degree two.

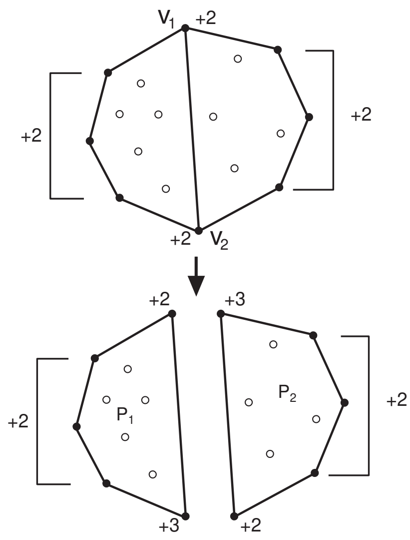

Figure 3: Partitioning the polygon P (the number at a vertex indicates its maximal load).

We then connect v_1 and v_2 by a line that divides P into two convex polygons P_1 and P_2 (see Fig. 3). Note that we are going to regard v_1 (v_2) as a vertex of load 3 with respect to P_2 (P_1). In this way we can assure that at most one additional edge is connected to both

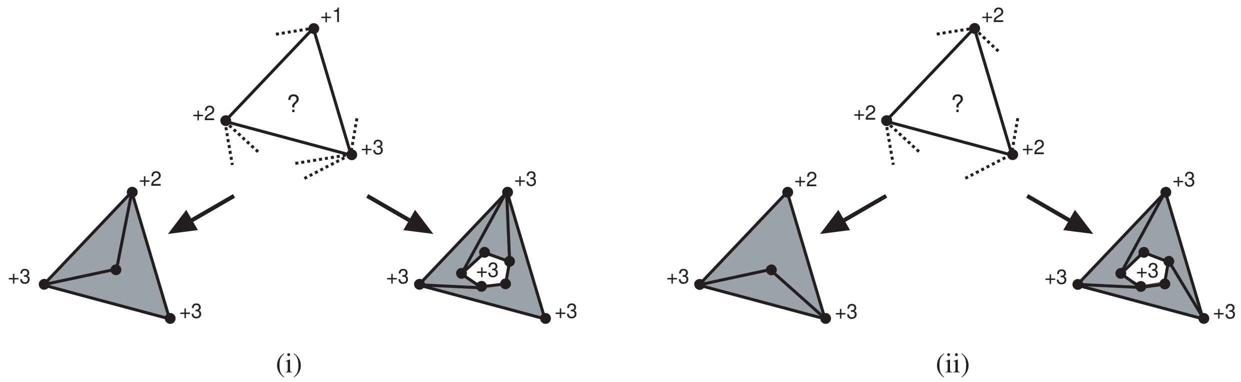

Figure 4: Shelling a triangle P (the number at a vertex indicates its maximal load).

and hence their final degree does not exceed five.

If P_1 (P_2) is a triangle then its load is at most six, since, in addition to v_1 and v_2, it contains only one more vertex on its boundary whose degree is at most three. Furthermore it contains at most one vertex of load 3, namely v_2 (v_1).

If P_1 (P_2) is a not a triangle, then we connect the two neighbours of v_2 (v_1) by a geodesic such that the resulting pseudo-triangle has no points in its interior (see Fig. 5 for an illustration of the operation applied to P_2).

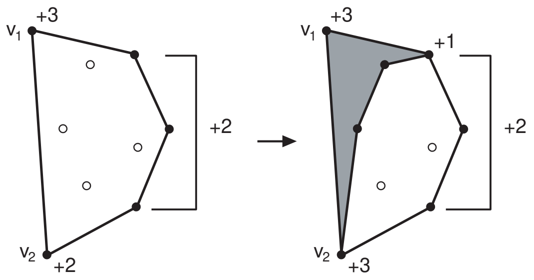

Figure 5: Cutting off v_1.

The remaining polygon P_1' (P_2') carries a load of at most six, where v_1 (v_2) contributes at most three, the geodesic contributes one more, and the at most two additional vertices of degree three contribute one each. Furthermore, P_1' (P_2') contains at most one vertex of load 3, namely v_1 (v_2).

2.2 Shell

Assume we are given a triangle P that satisfies both invariants. There are only two possible distributions for the maximum load on the vertices of P: one vertex of load one, two and three each (see Fig. 4(i)) or three vertices of load two (see Fig. 4(ii)).

If the interior of P is empty there is obviously nothing to do. If P contains one point in its interior, add two edges from this point to two vertices of load two or less (see the left subfigures of Fig. 4). Otherwise, compute the convex hull P' of the points in the interior of P. In the first case add two tangents from the vertex of load one to P' and one corresponding tangent from the vertex of load two (see Fig. 4(i)). In the second case draw one tangent from each vertex of P counter-clockwise to P' (see Fig. 4(ii)). Again, note that the shaded pseudo-triangles do not contain any points in their interior. In both cases P' carries a load of three and has at most one vertex of load three on its boundary.

2.3 Prune

Assume we are given a convex polygon P that satisfies both invariants, is not a triangle (i.e. P has at least four vertices on its boundary) and has vertices of load two or three (i.e. vertices of degree four or five).

We identify the vertex v_{max} of the highest load and connect its two neighbours by a geodesic such that the resulting pseudo-triangle does not contain any points in its interior. This geodesic now forms part of the boundary of P' – the high degree vertex v_{max} has been *pruned* (see Fig. 6).

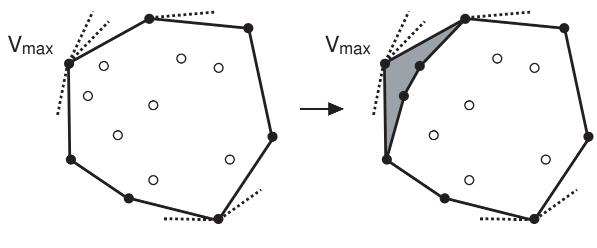

Figure 6: Pruning an high degree vertex.

The load of P' is $l(P') = l(P) - l(v_{max}) + 2 \leq l(P)$, since the new geodesic adds 2 to the load of P', but v_{max} is a vertex of degree at least four.

If $l(v_{max}) = 3$ then there was at most one vertex of degree four on the boundary of P, all other vertices must have been of degree three or less. This implies that at most one of v_{max} neighbours had degree four and therefore at most one vertex of degree five is created by the geodesic.

If $l(v_{max}) = 2$ then v_{max} is not necessarily uniquely defined. We can therefore choose v_{max} such that at least one of its neighbours has degree at most three. This is always possible, even if there are three vertices of degree four on the boundary of P, since P has at least four vertices. This again implies that at most one vertex of degree five is created by the geodesic.

Theorem 3 *This recursive construction results in a minimum pseudo-triangulation of vertex degree at most 5.*

3 Pseudo-triangulation of bounded face degree

We describe an iterative construction for a pseudo-triangulation for a set of n points in general position. First we form and triangulate the convex hull of the set of points. (Note that every triangulation is a minimum pseudo-triangulation for a set of points in convex position). Then we insert each internal point in turn into the pseudo-triangulation. To insert a point into a triangle we simply connect two arbitrary vertices of the respective triangle to the new vertex (see Fig. 7).

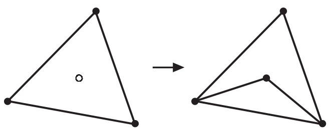

Figure 7: Inserting a vertex into a triangle.

To insert a point into a quadrilateral, we observe that the extensions of the edges that form the side of the pseudo-triangle that consists of two edges partition the pseudo-triangle into three regions (see Fig. 8). We choose two of the vertices of the pseudo-triangle depending on which region the points falls into and connect them to the new vertex (see Fig. 8).

Theorem 4 *Every set of points in the plane allows a pseudo-triangulation with pseudo-triangles that contain at most 4 vertices.*

4 Open problems

We would like to know if a bound on the vertex or face degree can be efficiently maintained as points move along linear trajectories.

5 Acknowledgments

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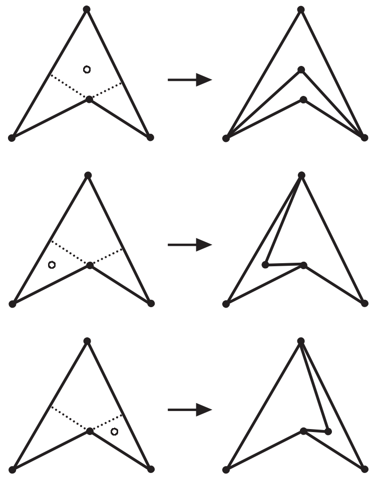

Figure 8: Inserting a vertex into a quadrilateral.

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