The Reflex-Free Hull*

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Abstract

We define a reflex-free hull in three dimensions as an intersection of reflex-free sets. The reflex-free hull allows a rich set of topological types, yet for polyhedral input with n edges, it remains a polyhedral set with $O(n)$ edges. This is in contrast to other possible hull definitions that give non-planar surfaces and higher combinatorial complexity. The reflex-free hull is related to filling cavities, as in the manufacturing process of casting, but we sketch examples to indicate that computing a reflex-free hull will be a challenging problem.

1 Introduction

Computational geometers have identified many classes of 2D polygons (convex, star-shaped, L-convex, externally visible, edge-visible, LR-visible, street, person... [4, 6]), but few classes of 3D polyhedra. Perhaps the fact that 3D polyhedra support rich classes of topological structure in the form of knots and links has overshadowed the identification of geometric structure. In applications such as manufacturing or molecular analysis, however, geometric structures such as mold cavities or docking sites are important.

In the plane, the difference between a simple polygon and its convex hull is a number of simple, polygonal bays, from which one can obtain a natural description of a polygon as a tree of unions and differences of convex pieces [5]. In space, it has been suggested that the same approach be used to define pockets in a search for casting directions [2], but in fact the difference between a polyhedron and its convex hull need not have a natural decomposition, and may have more complicated topology than the original polyhedron [1].

We propose a reflex-free hull, defined as an intersection of reflex-free sets. Section 2 defines the reflex-free sets and plane cavities, which may be seen as analogues to bays in polygons. Section 3 defines the reflex-free hull, *Rfh*, and Section 4 establishes some basic results about the *Rfh* of polyhedral sets, including the fact that the *Rfh* has linear complexity even though it allows a rich set of topological types. Section 5 shows that the reflex-free hull bounds the limit of a process of filling cavities, but that computing it in this manner would be challenging. Finally, Section 6 relates the reflex-free hull to other possible hull definitions that either have high complexities or limited topologies.

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Detailed proofs may be found in the full version of the paper.

2 Definitions

We begin by defining some notation. Our sets will be drawn from real three-dimensional space, $\mathbb{R}^3$, so the *complement* of a set $\overline{S} = \mathbb{R}^3 \setminus S$. We use the Euclidean metric, and denote the distance between two points by $d(p,q)$ and between two sets $d(A,B) = \limsup\{d(a,b) \mid a \in A, b \in B\}$.

At a point $p \in \mathbb{R}^3$, for any chosen scalar $\varepsilon > 0$ and vector $v \in \mathbb{R}^3$, we define the ε-*ball* $B_\varepsilon(p) = \{q \mid d(p,q) < \varepsilon\}$, the *v-halfspace* $h_v(p) = \{q \mid (q-p) \cdot v \leq 0\}$, and the *hemisphere* $H_{v,\varepsilon}(p) = B_\varepsilon(p) \cap h_v(p) \setminus \{p\}$. Note that a hemisphere is neither open nor closed—to simplify a later classification, $H_{v,\varepsilon}(p)$ does not contain p or the boundary points of $B_\varepsilon(p)$. Also, we may suppress the v and ε when they are understood from context.

For our purposes, a *polyhedron* is the union of the simplices in a finite simplical complex. A *polyhedral set* is homeomorphic to a polyhedron. We restrict our discussion to polyhedral sets to avoid wild topological beasties like the Alexander horned sphere [3]. Section 4 further restricts the discussion to polyhedra when it investigates combinatorial properties of reflex-free hulls.

The *interior* of a point set Q, denoted $\mathrm{int}(Q)$ consists of those points that have a neighborhood homeomorphic to a ball. All other points are on the *boundary*, $\mathrm{bd}(Q)$. We further classify each boundary point p based on hemispheres at p. Point $p \in \mathrm{bd}(Q)$ is a

- *reflex point* if there is a hemisphere at p inside Q. That is, if there exists a vector v and $\varepsilon > 0$ such that $H_{v,\varepsilon}(p) \subset \mathrm{int}(Q)$.
- *convex point* if there is a hemisphere outside Q. That is, if there exists $H_{v,\varepsilon}(p) \subset \overline{Q}$.
- *flat point* if there exists an $\varepsilon > 0$ and $v \in \mathcal{R}^3$ such that $H_{v,\varepsilon}(p) \subset Q$ and $H_{-v,\varepsilon}(p) \subset \mathrm{cl}(\overline{Q})$.
- *nearly reflex point* if p is neither reflex nor flat and there exists $H_{v,\varepsilon}(p) \subset Q$.
- *nearly convex point* if p is neither convex nor flat and there exists $H_{v,\varepsilon}(p) \subset \mathrm{cl}(\overline{Q})$.
- *saddle point* otherwise. That is, for every $\varepsilon > 0$ and v, hemisphere $H_{v,\varepsilon}(p)$ intersects both the interior and complement of Q.

For an example, we can classify points on the boundary of a three-manifold polyhedron. Points on faces are flat, points on edges are nearly reflex or nearly convex (or flat in the degenerate case of a dihedral angle of 180°), and points at vertices are convex, reflex, or saddle (except in degenerate cases of incident coplanar faces/edges). In a coffee mug, the reflex points are at the bottom of the bowl.

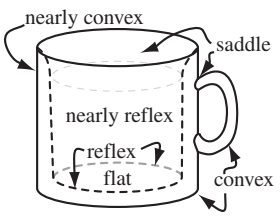

Figure 1: Classifying

For a polyhedral set Q and closed half-space h, we call a bounded connected component of $\overline{Q} \cap h$ a *plane-cavity*. If Q has a plane-cavity, then it has an infinite number of them.

Lemma 1 *A polyhedral set Q has a plane-cavity $\overline{Q} \cap h$ for some half-space h if and only if there is a reflex point in* $\mathrm{bd}(Q)$.

Proof: We can perturb the hemisphere defining a reflex point to make a plane-cavity, and we can find a bounding sphere of a plane-cavity to reveal a reflex point. □

We say that Q is *reflex free* if Q is a polyhedral set with no reflex points. By the previous lemma, a polyhedral set has no plane-cavities if and only if it is reflex free. In the full paper we also prove other basic properties.

3 The reflex-free hull

Define the *reflex-free hull* of a set Q as the intersection of all reflex-free sets that contain Q. For example, the reflex-free hull of a torus is itself; reflex-free hull of a coffee cup would fill the cup but preserve the handle. The reflex-free hull of a set of discrete points would be these points, because a union of balls around the points is reflex-free. The motivation for defining the reflex-free hull is to find a structure that surrounds a set, but fills in depressions or docking sites. We first show that the reflex-free hull of a set Q, denoted by $Rfh(Q)$, is well defined.

Lemma 2 *Let $\{Q_\alpha\}$ be a family of reflex-free sets. The intersection $\bigcap Q_\alpha$ is also reflex-free.*

Proof: Let $\mathcal{A} = \bigcap Q_\alpha$. Note that $\mathcal{A}$, as the intersection of closed 3-manifolds, is a closed set. Assume to the contrary that $\mathcal{A}$ has a reflex point p. Let $H_{v,\varepsilon}(p)$ be a hemisphere at p that is inside $\mathrm{int}(\mathcal{A})$. As in Lemma 1, there is a δ such that the translated plane $h_v(p+\delta v)$ bounds a plane-cavity X of $\overline{\mathcal{A}}$. But X cannot be contained in each reflex-free set Q_α, since p is a boundary point of $\mathcal{A}$. Thus, the plane $h_v(p+\delta v)$ must bound a plane-cavity of Q_α, which contradicts that Q_α is reflex-free. □

With this lemma, we can show that the reflex-free hull is well-defined.

Theorem 3 *For a closed set Q, the reflex-free hull $Rfh(Q)$ is reflex-free, and satisfies $Rfh(Rfh(Q)) = Rfh(Q)$.*

To develop the properties of reflex-free hull, we use sculpting operations that do not create reflex vertices. For example, drilling a hole through a reflex-free set keeps the set reflex-free. We define the ε-*tube* for a line segment s as $C_\varepsilon(s) = \{x \mid d(x,s) < \varepsilon\}$.

Lemma 4 *Given a reflex-free set Q, and a line segment s with endpoints in $\overline{Q}$, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \le \varepsilon_0$, $Q \setminus C_\varepsilon(s)$ is reflex-free.*

The reflex-free hull $Rfh(Q)$ inherits its convex, nearly-convex, and saddle points from the underlying set Q.

Lemma 5 *For a closed set Q, every convex, nearly-convex, or saddle point p of $Rfh(Q)$ is a point of Q. In particular, convex points of $Rfh(Q)$ are convex points of Q, nearly convex points of $Rfh(Q)$ are convex or nearly convex points of Q, and saddle points of $Rfh(Q)$ are convex, nearly convex or saddle points of Q.*

Proof: If there is such a point of $Rfh(Q)$ that is not on Q, we can apply a sculpting operation to remove it. □

4 The reflex-free hulls of polyhedra

In this section, we consider the reflex-free hull of a polyhedron. We define the *size* of a polyhedron to be the number of vertices, edges, and faces on its boundary. We show that the reflex-free hull of a polyhedron is a polyhedron of the same asymptotic size. We find this surprising in light of the high complexity of other definitions of hulls that we sketch in the next section.

We sketch the proof, which is in a couple of stages.

Lemma 6 *The reflex-free hull of a polyhedron $\mathcal{P}$ of size n is a polyhedron with $O(n)$ vertices.*

Proof: This proof is actually divided into a couple of technical lemmas. First, we show that the nearly convex and nearly reflex points can be partitioned into a maximal set of line segments. Then we show that the boundary of $Rfh(\mathcal{P})$ is a set of polygonal faces, so that $Rfh(\mathcal{P})$ is polyhedral if $Rfh(\mathcal{P})$ has finite size.

The vertices of $Rfh(\mathcal{P})$ are convex or saddle points of $Rfh(\mathcal{P})$, and these are inherited from convex, nearly convex, nearly reflex, and saddle points of $\mathcal{P}$. The number of convex vertices of $Rfh(\mathcal{P})$ is bounded. We bound the number of saddle points by observing that they come either from vertices of $\mathcal{P}$ or from reflex edges of $\mathcal{P}$. Showing that each

edge can give rise to at most two saddle points completes the proof. □

Now we show that for a polyhedral set $\mathcal{P}$ the reflex-free hull $Rfh(\mathcal{P})$ is a polyhedral set with the same asymptotic size as $\mathcal{P}$.

Theorem 7 *The reflex-free hull of a polyhedron of size n is a polyhedron whose size is $O(n)$.*

Proof: From Lemma 6 we know that $Rfh(\mathcal{P})$ is polyhedral and it has $O(n)$ vertices, where n is the size of $\mathcal{P}$. We first bound the genus of $Rfh(\mathcal{P})$ using a curvature argument, which shows that the increase in genus from $\mathcal{P}$ to $Rfh(\mathcal{P})$ is $O(n)$. We have to deal with one complication of pinched vertices. From Euler-Poincare formula, we know that $V-E+F=2-2g$, where V, E, F, and g are the number of vertices, edges, faces, and genus of $Rfh(\mathcal{P})$. Since $3F \leq 2E$, we deduce that $V-E/3 \geq 2-2g$ and so $E \leq 3V+6g-6$ which is $O(n)$. This completes the proof of the theorem. □

5 The reflex-free hull and cavities

Recall that for a closed 3-manifold Q and halfspace h, we defined a *plane cavity* as a connected component of $h \cap \overline{Q}$.

If we iteratively fill up plane cavities, then we obtain a sequence of interesting sets. We describe this process precisely as follows. Let $\mathcal{P}_0$ denote $\mathcal{P}$. Given some plane-cavity $\mathcal{C}_0$ of $\mathcal{P}_0$, we form the union $\mathcal{P}_0 \cup \mathcal{C}_0$ to obtain a new polyhedron $\mathcal{P}_1$. We repeat this process an arbitrary number of times. With respect to any sequence σ_k of k plane cavities, we denote the above process by $(\mathcal{P}, \sigma_k)$ and we call a connected component of $\mathcal{P}_k \setminus \mathcal{P}$ a *cavity* of $(\mathcal{P}, \sigma_k)$. A face f of $\mathcal{P}$ *bounds* a cavity $\mathcal{F}$ of $(\mathcal{P}, \sigma_k)$ if f lies partially or completely on the boundary of the closure of $\mathcal{F}$.

We conjecture that in the limit, we may obtain the reflex-free hull by filling cavities. Equivalently, for a closed 3-manifold $\mathcal{P}$, the cavities are $Rfh(\mathcal{P}) \setminus \mathcal{P}$. The difficulty is the usual one of proving that the limit exists. We can show

Theorem 8 *For a closed polyhedron $\mathcal{P}$, the limit of the process of filling cavities is a subset of the reflex-free hull, $Rfh(\mathcal{P})$.*

Proof: We show by induction that the cavities identified by the filling process are inside all reflex-free sets that contain $\mathcal{P}$. Specifically, we prove that any polyhedral set Q that contains $\mathcal{P}$, but does not contain $\mathcal{P}_k$, has a reflex point. □

Unfortunately, we do not know how to turn this definition into an efficient procedure to compute the reflex-free hull of a polyhedron. The process of filling in one reflex vertex can create others at reflex edges. Figure 2 illustrates one example in which filling cavities must be taken to the limit to attain the reflex-free hull.

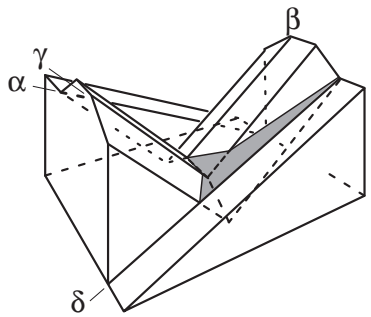

Figure 2: Each filling step creates reflex and saddle points

Start with the cube $[-5,5]^3$ and subtract the following sets: $\{(x,y,z) \mid x \in [-1,1], z > |y|\}$, $\{(x,y,z) \mid |x| \in [1,3], z > |y|-|x|+1\}$, $\{(x,y,z) \mid x \in [3,5], z > y/2+1\}$, and $\{(x,y,z) \mid x \in [-3,-5], z > -y/2+1\}$, to obtain an object illustrated in Figure 2. There are four labeled lines that are relevant in this example. We parameterize them by z. Two are *pivots*, $\alpha(z) = \{(-3, 2z-2, z)\}$ and $\delta(z) = \{(3, 2-2z, z)\}$, and two lines end at saddle points, $\beta(z) = \{(-1, z, z)\}$, and $\gamma(z) = \{(1, -z, z)\}$. With a little algebra, we observe that a plane that contains δ and intersects β at $z = t$ must intersect γ at $z = (t+2)/6$. By symmetry, the plane through α and $(1, -t, t)$ intersects β at $z = (t+2)/6$.

Initially, there are two reflex vertices with coordinates $(\pm 3, 0, -3)$. We can eliminate the first by filling the cavity defined by the plane through pivot line δ and the saddle point $\beta(0)$; this plane intersects γ at $z_1 = 2/6$. We eliminate the second by filling to the plane through α and the newly created saddle point $\gamma(z_1)$; this plane intersects β at $z_2 = 7/18$, and creates a new reflex vertex where the two filling planes meet. From now on, we fill from a pivot line to a saddle at $z_i = (z_{i-1}+2)/6$. The reflex-free hull for this example has a reflex edge along the line through $(-1, 2/5, 2/5)$ and $(1, -2/5, 2/5)$, which happens to be the unique line incident to α, β, γ, and δ. Thus, we approach, but never reach the reflex-free hull.

If reflex edges incident on four polyhedron edges were the worst that could occur, we could still hope for a polynomial-time algorithm for the reflex-free hull by inspecting all 4-tuples of edges to see if they support a common line. Unfortunately, however, reflex edges may be defined by lines that hit only two polyhedron edges. Figure 3 shows a schematic of such an example consisting of eight spheres, which could be approximated by polyhedra. In it, we have a sequence of eight geodesic triangles that share the thick segments, which are reflex edges of the hull. The extensions of reflex edges (dash-dotted) intersect within the incident geodesic triangles. Thus, the plane equations for the triangles on the reflex hull all depend on each other. The reader who would like to find an algorithm to compute reflex-free hulls is advised to build similar examples from modeling clay.

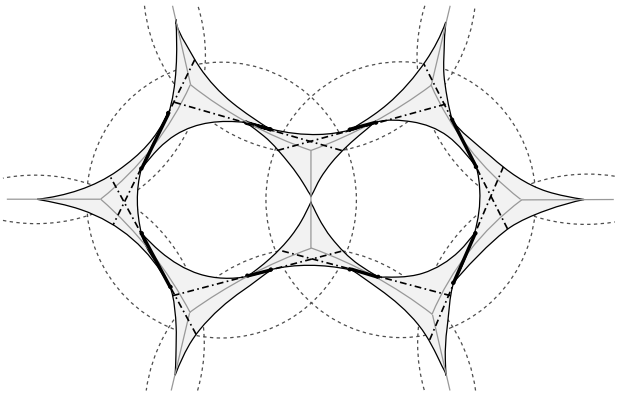

Figure 3: Part of the reflex hull of an appropriately placed set of eight spheres. The planes determining the new boundaries of this hull are defined by a sequence of saddles in such a way that if any saddle is moved then all plane equations change.

6 Other hulls

The fact that the reflex-free hull has linear complexity may not at first seem surprising. In this section, we see other natural definitions for hulls with far worse complexities.

For a closed set S, we may obtain the convex hull $CH(S)$ by removing halfspaces do not intersect S. We may obtain the reflex-free hull $Rfh(S)$ as the intersection of reflex-free sets that contain S. In a similar way, we may define the *line hull*, $LH(S)$, by removing lines that do not intersect S, or, more formally, as the intersection of sets containing S that are the complements of lines. We can order the hulls of a set S by inclusion: $S \subseteq Rfh(S) \subseteq LH(S) \subseteq CH(S)$. In general, all inclusions are strict.

In the plane, the line hull of a connected set is the same as its convex hull, but the line hull of a disconnected polyhedral set of size n may can have $\Theta(n^4)$ complexity, as it is related to an arrangement of the $\Theta(n^2)$ lines tangent to pairs of vertices of S. In $\mathbb{R}^3$ the line hull is bounded by pieces of ruled surfaces, including hyperboloids. Sergei Bespamyatnikh, in private communication, described an example of a connected polyhedral set S of size n whose line hull has $\Theta(n^9)$ complexity. Begin with the six faces of a large, axis-aligned cube, and cut a small square hole in the center of each. Near the center of this cube we have three families of lines, each roughly parallel to one of the three axes. Block the lines parallel to the x axis with three squares parallel to the yz plane, and cut n parallel slits in different directions each square so that the lines that do pass through these slits form $\Omega(n^3)$ hyperboloids near the center of the cube. Repeat this for the y and z axes, so that these hyperboloids have $\Omega(n^9)$ intersections.

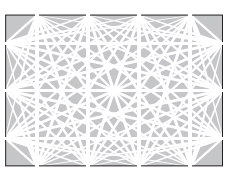

Figure 4: Line hull of 18 black segments

Some other natural hull definitions suffer from similar complexities. One could define the *star hull*, $SH(S)$, as the intersection of all star-shaped polyhedra that contain S. Each point p that is not in the star hull is excluded from some star-shaped polyhedron, which says that p has a ray to infinity that does not intersect the interior of S. Thus, one could define the *ray hull* as the intersection of sets containing S that are the complements of open rays. We say that a point p of a closed set X is *externally visible* if there is a ray from p that does not intersect the interior of X. A set X is *externally visible* if every point on its boundary is externally visible. Thus, one could define the *externally visible hull* of a closed set S to be the intersection of all externally visible sets that contain S.

It is not difficult to see that the star hull, ray hull, and externally visible hull are identical, and that $S \subseteq SH(S) \subseteq LH(S)$, with strict inclusion for many sets S. The reflex free hull and star hull cannot always be ordered by inclusion: The boundary of the star hull contains any reflex vertices from S that are externally visible. The boundary of the reflex free hull may contain points that are not externally visible, as can be seen in an example of nested tori rotated about a common axis so they form a spherical shell.

A minor modification of Bespamyatnikh's construction shows that the star hull can again be bounded by hyperboloids, and may have $\Omega(n^9)$ complexity.

Acknowledgments

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