Separating Several Point Sets in the Plane *

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Abstract

In this paper we study some problems on the separability of k disjoint point sets in the plane. On one hand, we give algorithms for finding minimum-cardinality separators by means of parallel lines or rays with common apex. On the other hand we show how to decide whether it is possible to separate by $k-1$ parallel lines, k rays with same origin, an arrangement of 2-3 lines or, in the projective sense, by k lines through a point.

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1 Introduction

Let $C_1, \ldots, C_k$ be finite disjoint point sets in the plane; we refer to C_i as the set of points of *color* i. We denote $n_i = |C_i|$ and $n = n_1 + \cdots + n_k$. A finite set S of curves in the plane is a *separator* for the sets $C_1, \ldots, C_k$ if every connected component in $\mathbb{R}^2 - S$ contains objects only from some C_i. We also say that each connected component is *monochromatic*.

A geometric separator is useful in fields in which discrimination-classification is required, and can also be defined for objects other than points and in higher dimensions. Deciding the existence of a particular class of separator, or optimizing within a class when several possibilities are available, are the issues that have attracted most attention.

The case $k = 2$ (commonly a *red set* and a *blue set*) has been largely studied. A separating line, if there is any, can be found in linear time [12]. When line separability is not possible, one may consider to find an arrangement of the fewest lines decomposing the plane into monochromatic cells [6] (the *shattering problem*) or to find a polygonal separator with as few edges as possible [5]) (the *minimum-link red-blue separation problem*). These two variants are known to be NP-complete. Another possibility is to consider *wedges* and *strips*: A wedge is the union of two rays with common origin (the *apex*); a strip is the union of two parallel lines; their common slope is *the slope of the strip*. A *double wedge* is the union of two crossing lines. Optimal algorithms for deciding whether such separators exist are described in [1, 2, 8, 9], where some additional optimization issues are also considered.

In this paper we consider the separation problem for $k > 2$ sets. Our results include algorithms for finding minimum-cardinality separators by means of parallel lines and by means of rays having a common apex. We also provide efficient algorithms for deciding the existence of some specific separators: $k-1$ parallel lines, k rays with common origin, k lines through a point (where the cells are taken in the projective sense), and by two or three lines (for $2 \leq k \leq 4$ and $2 \leq k \leq 7$ respectively), i.e., shattering the sets of points in monochromatic regions by two or three lines.

2 Separating with parallel lines

Given $C_1, \ldots, C_k$ disjoint colored sets of points in the plane, at least $k-1$ parallel lines are required in order to separate the sets in monochromatic parallel strips. If the slope of the parallel lines is given, the problem is trivial: in $O(n \log n)$ time we project orthogonally the points onto a line perpendicular to the given

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direction and we count the number of color changes of the projected points. The problem becomes more interesting when the slope is not given, which we consider next.

Theorem 2.1. *The minimum number of parallel lines separating the k colored sets $C_1, \ldots, C_k$ in monochromatic strips, as well as the slope intervals of all the possible solutions, can be computed in $O(n^2 \log n)$ time.*

The complexity above is related to the open problem of deciding whether the intersection points of an arrangement of n lines in the plane can be sorted in x-order from left to right in $O(n^2)$ time. This problem is at least as hard as the classical problem of sorting $X + Y$, which also remains open.

For k colored sets of points we need at least $k - 1$ parallel lines to separate the points in monochromatic parallel strips. Now we consider the problem of deciding whether $k - 1$ parallel lines are enough.

Theorem 2.2. *Let $C_1, \ldots, C_k$ be disjoint colored sets of n points in the plane. Deciding whether the k sets are separable by exactly $k-1$ parallel lines, as well as computing the slope intervals of all the solutions, can be done in $O(kn) + O(k^2 \log k)$ time.*

This problem is closely related to the *disjoint projection problem* or *shadow problem* originally attributed to Preparata [11]: *Given n polygonal objects in $\mathbb{R}^2$, find a direction such that the set of objects projects disjoint orthogonally onto a line in this direction.* This problem has been solved in time $O(n^2 \log n)$ assuming constant size objects, i.e., with $O(1)$ vertices per object [3], but it is open whether it can be solved in time $O(n^2)$.

For k constant, the $k-1$ parallel lines separability of k disjoint sets of points has a $\Theta(n)$ time bound. It contrasts with the $\Theta(n \log n)$ time lower bound for the strip separability problem of two disjoint sets, i.e., two parallel lines separating two point sets (see [2, 8]). The difference is that while in the strip separability of two disjoint point sets we may have points of the same color in both sides of the strip, now each colored set lies in only one strip or one half plane. In fact, it can be proved, using similar arguments as in [2], that the separability of k disjoint colored sets of points by k parallel lines has an $\Omega(n \log n)$ lower bound (as a corollary of Theorem 3.4).

3 Sector separability

In the *sector separability* (or *radial separability*) problem we want to have the k disjoint colored sets of points lying in monochromatic wedges defined by rays emanating from a common point, *the apex*. Observe that this kind of separability is always possible if many rays are used. Deciding whether two disjoint point sets are wedge separable (i. e., using two rays) and to compute the locus of the apices of the rays can be done in $O(n \log n)$ as shown in [8]. In general, the most interesting issue is to determine the apex which minimizes the number of required rays used, this is the problem we consider next.

Theorem 3.1. *Let $C_1, \ldots, C_k$ be disjoint colored sets of total n points in the plane. A minimum-cardinality set of rays (having common apex) separating the sets can be computed in $O(n^4)$ time.*

A restricted version of the problem arises by constraining the apices to lie on a given line.

Theorem 3.2. *Let $C_1, \ldots, C_k$ be disjoint colored sets of total n points in the plane and let l be a line. A minimum-cardinality set of rays separating $C_1, \ldots, C_k$, with common apex on l, can be computed in $O(n^2 \log n)$ time. The locus of possible placements of the apex can be obtained within the same time bound.*

In order to separate k colored sets of points into monochromatic sectors we need at least k rays from a common apex. Deciding whether exactly k rays are enough is the problem that we consider next.

Theorem 3.3. *Let $C_1, \ldots, C_k$ be disjoint colored sets of points in the plane. Deciding whether the sets are separable by k rays with common apex can be done in $O(n \log n) + O(nk^4 \log k)$ time. The locus of the possible placements of the apex on l can be obtained within the same time bound.*

The constructions in [2] can be adapted to prove the following.

Theorem 3.4. *Deciding in the algebraic decision tree model whether k disjoint colored sets of points are separable by k rays with a common apex requires $\Omega(n \log n)$ operations.*

We can simplify the problem of the k rays separability by restricting the apex of the rays to lie on a given line.

Theorem 3.5. *Let $C_1, \ldots, C_k$ be disjoint colored sets of points in the plane and let l be a line. Deciding whether the sets are separable by k rays from a common apex on l can be done in $O(n \log n) + O(k^2 \log k)$ time. The locus of the possible placements of the apex on l can be obtained within the same time bound.*

4 Projective separation by lines through a point

An $O(n \log n)$ algorithm for deciding the double wedge separability of two disjoint sets of points, as well as for computing the region of the plane formed by the apices of the separating double wedges, is described in [9], and a matching lower bound is given in [2]. Since opposite quadrants would correspond to the same projective cell, double wedge separability can be considered as separation by two lines through a point, *the apex*, in a projective sense. We next consider the separability of k sets by k lines through a point in this same sense.

Definition 4.1. *Given k disjoint colored sets of points, we say that they are separable by k lines through a point in the projective sense, k-projective separable for short, if there exist k lines through a point p, the* projective apex, *such that opposite sectors are monochromatic.*

Additional constraint. We consider here only the k-projective separability with the additional constraint that opposite sectors are not empty. This implies that the sets are pairwise separable by double wedges with common apex in p.

Theorem 4.1. *Let $C_1, \ldots, C_k$ be disjoint colored sets of points in the plane. Deciding whether they are k-projective separable with the additional constraint can be done in $O(kn \log n)$ time. The locus of all the possible placements of the projective apices can be obtained within the same time bound.*

The k-projective separability without the additional constraint is under consideration. The constructions in [2] can be adapted to prove the following.

Theorem 4.2. *Let $C_1, \ldots, C_k$ be disjoint colored sets of total n points in the plane. Deciding whether they are k-projective separable with the additional constraint requires $\Omega(n \log n)$ time.*

5 Separating with two or three lines

In this section we discuss separability by means of two or three lines. The brute force approach would give $O(n^4)$ and $O(n^6)$ algorithms for two and three lines, respectively, due to the degrees of freedom. We only consider the case in which the n points are in general position. First, we study the separability by two lines of k colored sets of points, $2 \leq k \leq 4$, having as a special possibility the case that the lines are parallel. Some of the cases are just known results already solved in [2, 8, 9]. We have the following theorem.

Theorem 5.1. *Given k colored sets of total n points, $2 \leq k \leq 4$, the separability of the k sets by two lines can be decided in $O(n \log n)$ time.*

For separability by three lines of k colored point sets, $2 \leq k \leq 7$, we again consider first the situation in which the three lines are parallel. In all other cases several possibilities arise, and we always assume that the sets are not line separable, wedge separable or double wedge separable. The cases are classified depending on the number and kind of regions of each color. In some cases we use results form [2, 4].

Theorem 5.2. *Given k colored sets of total n points, $2 \leq k \leq 7$, the separability of the k sets by three lines can be decided in $O(n^3 \log n)$ time.*

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