Narrowing Light Rays with Mirrors

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Abstract

Pach raised the question of whether a match lit in a finite forest of disjoint circular mirror trees necessarily sends light to infinity [CFG90]. In other words, can one *trap* light rays from a point source with such mirrors? This remains an unsolved problem. We pose related mirror trapping questions. Can one trap light with arbitrarily curved mirrors? Here the answer is YES. Can one trap light with a finite number of disjoint, double-sided segment mirrors? Despite the naturalness of this question, it seems unexplored. We were unable to settle it, but can state some partial results, and formulate some conjectures.

Because these questions appear difficult, we explore whether the rays from a point source can be *narrowed* by a collection of mirrors, leaving all rays that escape to infinity with nearly the same direction, in the sense that the angle between any pair is less than $\epsilon > 0$. Here we prove the answer is YES for both segment mirrors and for circular mirrors. The constructions are rather different, and in the case of circular mirrors, a bit delicate.

1 Illuminating Polygons

Klee asked whether every polygonal room whose walls are mirrors (we'll call this a *mirror polygon*) is "illuminable" from every interior point [Kle69, KW91], i.e., does a point source of light illuminate every point of the polygon? Twenty-five years later, Tokarsky answered NO by constructing rooms that leave one point dark when the light source is located at a particular spot [Tok95] [Ste96]. We feel that interesting questions remain. For example:

Conjecture 1 *The measure of the set of dark points in any mirror polygon with one interior point source is zero.*

The reason we believe this conjecture is that there is a sense in which Tokarsky's dark points derive directly from the polygon vertices. Although it is clear how light rays should bounce off an interior point of a mirror, it is less clear what should happen at a vertex. Tokarsky used the natural model that a light ray dies when it hits a vertex; call this the *vertex-kill* model. One can view his constructions as cleverly parlaying this property to arrange for an interior dark point, for his dark points may only be reached by rays through vertices. There are several other possible models for ray-vertex interaction. The *round-vertex* model assumes each vertex is represented by a circular arc connecting the incident edges, with behavior determined by the limit as the circle radius approaches 0. Thus a ray hitting a convex vertex bounces straight back, reversing its approach path. The *butt-vertex* model cuts off each vertex with a segment perpendicular to the angle bisector, with behavior determined as the segment length approaches 0. A ray hitting a convex vertex in this model reflects across the angle bisector. These two models share a weakness, in that not all rays approaching a reflex vertex are reversible. This motivates the *split-ray* model, in which a ray approaching a reflex vertex from the middle "inner cone" splits to two rays, with each bouncing off the incident edges as if it hit those in their interiors. In several of Tokarsky's examples, the formerly dark points become illuminated under some of these models. Although we have not completely investigated these issues, the failure of Tokarsky's rooms suggests three conjectures:

Conjectures 2,3,4 *Under the round-vertex, butt-vertex, and split-ray models, every mirror polygon is illuminable from every interior point.*

We will return to mirror polygons in Section 2.3.

2 Trapping Rays with Mirrors

An interesting class of problems is suggested by that posed by Pach [CFG90, p.19] [Ste96]: If a match is lit in an "enchanted forest"[1] of mirror trees, must some light escape to infinity? More generally, let M be a finite set of disjoint mirrors; we will consider different restrictions on the shapes of the mirrors. Call a light ray from a source s *trapped* by M if the ray includes no point strictly exterior to the convex hull of M, and say a ray *escapes* otherwise. Can an M be found to trap every ray from some point source s (disjoint from M), or must some ray escape for any M and s?

We consider three shapes of mirrors:

1. Smooth, open curves of otherwise arbitrary shape.
2. Circles.
3. Segments.

2.1 Arbitrary Mirrors

The question of trapping a beam of parallel light rays was posed by Connett [Con92] and answered shortly

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[1] Terminology due to Alejandro López-Ortiz.

thereafter by constructions piecing together sections of ellipses and parabolas [Fra94, Pei94]. As it seems difficult to contruct a similar trap for a more diffuse "beam" of light, we instead employ half of Klee's smooth unilluminable room [KW91, Fig. 1.2]: see Fig. 1. Light rays the source x remain outside the ellipse foci a and a', and so never cross the segment aa'. Therefore light from a point source can be trapped by an open, smooth curve.

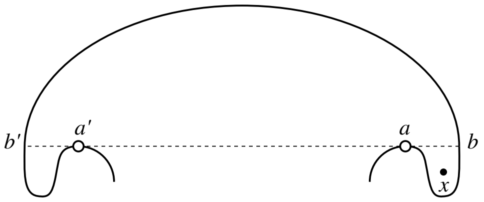

Figure 1: Light from the source x is trapped.

2.2 Circular Mirrors

Restricting the mirrors to be circles is Pach's problem. Our explorations (e.g., Fig. 2)[2] lead us to conjecture an answer to Pach's question:

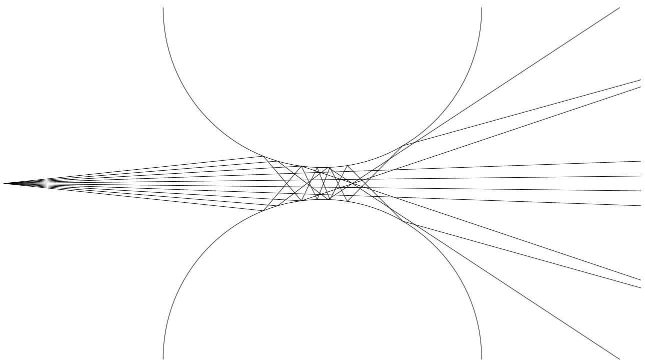

Figure 2: Angular spreading effect of disk mirrors.

Conjecture 5 *No finite collection of disjoint circular mirrors can trap all the rays from a point light source.*

2.3 Segment Mirrors

Strangely, the natural and at least superficially easier question of whether disjoint (two-sided) segment mirrors can trap light seems not to have been explored.[3] Even the extensive work falling under the rubric of "billiards in polygons" (e.g., [Tab95]) seems not to suffice to answer the question. Here we present a few partial results and risk some conjectures.

[2]This figure was produced by software we wrote to explore mirror questions experimentally. It is available at `http://cs.smith.edu/~orourke/Mirrors/`.

[3]Posed in a group conversation during CCCG '00.

Call the trajectory of a ray *periodic* if it rejoins itself after a finite number of mirror reflections.[4] Every periodic ray is trapped. Let the n mirror segments make angles μ_i, measured counterclockwise from the positive x-axis, $i = 1, \ldots, n$.

Lemma 1 *A set of mirrors supports periodic rays iff*

$$\mu_1 - \mu_2 + \cdots + (-1)^{n+1}\mu_n \equiv \pi (\mathrm{mod} 2\pi) \qquad (1)$$

Lemma 2 *The number of rays emitted by a source that fall into periodic trajectories is countable.*

Proof: Suppose a light ray from source s at angle ρ is periodic, hitting mirrors $i_1, i_2, \ldots, i_k$ in that order, and then repeating. Let M_i be the reflection matrix for mirror i, which reflects points in the line containing the mirror. Then s must map to its reflected image s':

$$s' = M_{i_k} \circ \cdots \circ M_{i_2} \circ M_{i_1} s \qquad (2)$$

With the hit sequence fixed, s' is determined, and s and s' determine ρ. So a given hit sequence determines a unique periodic ray.

There are $k!$ possible hit sequences of length k. Thus, over all k, there are only a countable number of possible hit sequences, and the claim follows. □

Although there are only a countable number of periodic rays, a theorem of Masur [Mas86] implies that at least sometimes[5] the set of directions leading to periodic orbits is dense. The reason Lemma 2 does not settle our question is that there might be nonperiodic but nevertheless trapped rays.

A deep result of Galperin et al. [GKT95] implies the closure of the points illuminated by a nonperiodic trapped ray must include at least one vertex (i.e., mirror endpoint). They proved this for mirror polygons but we can transfer it to our context. However, this does not seem to be enough to draw a definite conclusion for our problem. We can only employ their work to draw a weak conclusion in what is for us a rather unnatural model, as follows. Suppose surrounding each mirror endpoint is a light-absorbing disk of radius $\epsilon > 0$; perhaps a rubber bumper. This corresponds to what Delman and Galperin [DG01] call "billiards with pockets." Employing their result we can prove that, because the closure of a path includes a vertex:

Lemma 3 *If each mirror endpoint is surrounded by a finite-radius light-absorbing disk, then no collection of mirrors can trap a nonperiodic ray.*

[4]See [DO01] for an open problem concerning periodic rays in triangles.

[5]For mirrors that can form a "rational polygon."

It remains unclear if the ϵ-disks can absorb all the light rays. Rather than bumbers, it is more natural to use one of the vertex-reflection models considered earlier, e.g., the vertex-kill model. Then approaching a vertex arbitrarily closely does not suffice to kill the ray. We are left only with three increasingly strong conjectures:

Conjecture 6 *A collection of segment mirrors may trap only a countable number of nonperiodic rays from one source.*

For randomly-oriented mirrors, a random ray seems to escape with probability 1; see Fig. 3. Without exact computation, however, it is difficult to interpret this as evidence supporting the above conjecture.

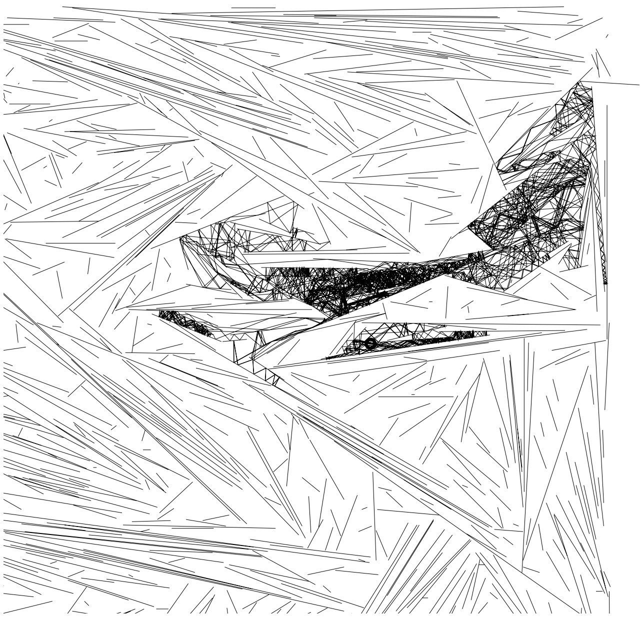

Figure 3: $n = 1000$ randomly oriented mirrors. Ray escapes (upper right) after 2941 reflections.

Conjecture 7 *A collection of segment mirrors may trap only a finite number of nonperiodic rays from one source.*

The above corresponds to the previously mentioned result for polygons with pockets.

Conjecture 8 *A collection of segment mirrors cannot trap any nonperiodic rays.*

Perhaps this last conjecture is too strong, but we offer it to spur search for a counterexample. Needless to say, establishing any one of these conjectures would imply an answer to our question:

Conjecture 9 *No collection of segment mirrors can trap all the light from one source.*

3 Narrowing Rays

Without firm answers to the natural trapping questions, we turn instead to the question of whether the rays from a point source can be narrowed by a collection of disjoint mirrors.[6] We say the rays are *narrowed* if the angle between any pair of rays that escape to infinity is less than $\epsilon > 0$. Here we prove that both segment mirrors and circular mirrors can narrow the light from a source in this sense.

3.1 Segment Mirrors

Theorem 4 *Given any $\epsilon > 0$, the light emitted by a point source can be narrowed to rays whose angles differs by less than ϵ by placement of a finite number of disjoint segment mirrors.*

Proof: *(Sketch).* We first surrounding the source s with three segments to reduce the light to three $90°$ near-cones, as shown in Fig. 4. Then each conical beam

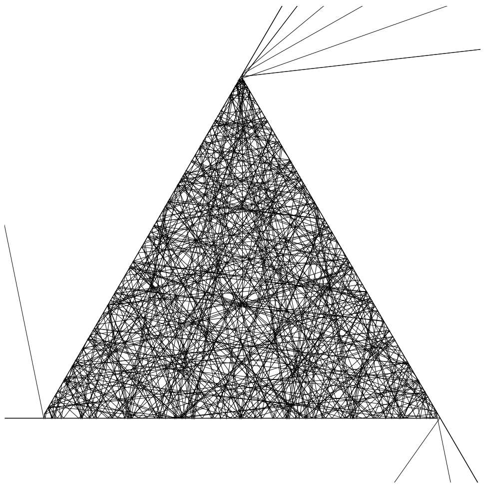

Figure 4: The source at the triangle centroid emits 10 rays. The few rays that have escaped through the small gaps would form a nearly conical beam of light without discretization.

is narrowed, following the halving strategy illustrated in Fig. 5. □

3.2 Circular Mirrors

Theorem 5 *Given any $\epsilon > 0$, the light emitted by a point source can be narrowed to rays whose angles differs by less than ϵ by placement of a finite number of disjoint mirror disks.*

Let the source s be at the origin. Place one disk very close to s below the x-axis, so that light is (nearly) confined to the upper halfplane. The main idea, due

[6] Posed in the same group conversation during CCCG '00.

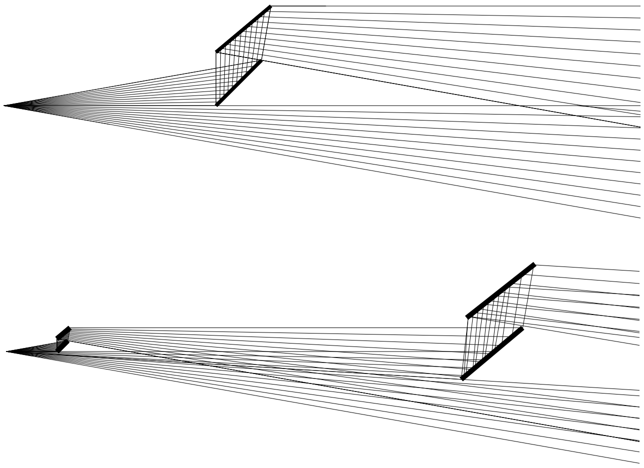

Figure 5: Above: 20° beam narrowed by half; Below: narrowed again, by nearly half, resulting in an approximate 5° beam.

to Therese Biedl, is to construct a "necklace of disks" which curl around in such a way that they reflect all the remaining light to the left. Let the circles be $C_1, C_2, \ldots, C_n$. The construction arranges that light from s tangent to C_i reflects off of C_{i+1} exactly horizontally. Fig. 6 illustrates the first 7 disks of one possible construction. Here each additional disk narrows the light about 10°. It becomes less clear what happens as the necklace curls into the second quadrant. Fig. 7 hints at how the construction is continued.

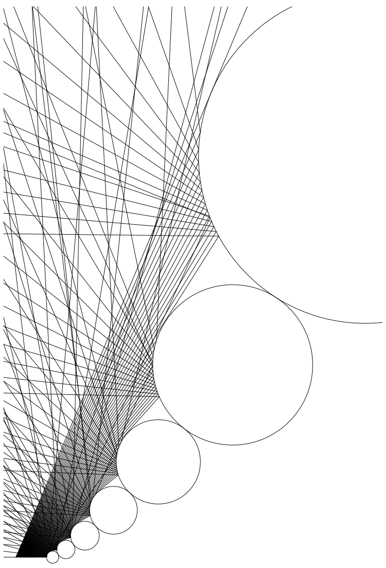

Figure 6: Necklace of mirror disks: first 7 disks.

With a necklace of $n = 24$ disks, our construction blocks more than 178° of the upper-halfplane light, thereby achieving an ϵ of less than 2°. The last disk in this series (not illustrated) has a radius of 10^{37}. Note that, although the rays are narrowed angularly, they

Figure 7: Necklace of mirror disks to disk 13 (the large, clipped disk). The first 10 disks are invisibly small near the origin; disks 11 and 12 are visible nestled under disk 13.

are not confined tightly spatially.

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