On the Number of Triangulations Every Planar Point Set Must Have*

(Extended Abstract)

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Abstract

We show that the number of straight line triangulations exhibited by any set of n points in general position in the plane is bounded from below by $\Omega((2+\varepsilon)^n)$ for some $\varepsilon > 0$. To the knowledge of the authors this is the first non-trivial lower bound.

1 Introduction

Efficiently counting the number of straight line triangulations of a given set of n points in the plane is an intriguing open problem. The currently fastest method is based on the recent concept of so-called triangulation paths [1]. But still the running time shows exponential growth and computations are limited to $n \leq 30$ for general sets. Moreover, no tight asymptotic upper and lower bounds are known. Best examples for maximizing this number give $2^{3n-\Theta(\log(n))}$ triangulations, whereas the currently best upper bound is much larger, although it has recently been improved from approx. 2^{8n} [5] to $59^{n-\Theta(\log(n))}$, see [9]. On minimizing this number there exists an example with $(\sqrt{12})^{n-\Theta(\log(n))}$ triangulations, see Section 3 for a brief discussion. Surprisingly no good general lower bound is known, although it is 'common knowledge' that there are 'always exponentially many' triangulations.

In this paper we quantify this 'common knowledge'. More precisely we consider the problem of finding a function $t(n)$ such that every set of n points in the plane[1] exhibits $\Omega(t(n))$ triangulations. In [6] it is shown that any triangulation on n points contains at least $(n-4)/6$ edges that can be flipped simultaneously. This immediately yields a first lower bound $t(n) = 2^{(n-4)/6}$.

We will present a general scheme based on induction, leading to a function $t(n) = c \cdot \tau^n$ for constants c and τ. The best values achieved up to now are $\tau = 2+\varepsilon$ for $\varepsilon \approx 0.012967$ and $c = 0.082263$. The main theorem of this paper can be stated as follows.

Theorem 1 *Let $t(n)$ denote the number of straight line triangulations any set of n points in general position in the plane exhibits. Then $t(n) \geq 0.0822 \cdot 2.0129^n$.*

The proof of the theorem will be given in the remaining sections. To the knowledge of the authors this constitutes the first non-obvious lower bound.

Throughout, let S be a set of n points in the plane. With $E(S)$ we denote the set of straight line segments with endpoints in S, i.e. the edge set of the complete geometric graph K_n of S. Two edges of $E(S)$ are said to cross each other if they properly intersect in their interior.

2 Recurrence Relations

Let $e \in E(S)$ be a segment spanned by S such that the supporting line of e splits S into two parts S_1 and S_2 of cardinality n_1 and n_2, respectively, $n_1, n_2 \geq 3$, see Figure 1 (a). We count the two endpoints of e for both subsets, thus $n_1 + n_2 = n+2$. Let $t(S)|_e$ be the number of triangulations of S containing e. Obviously $t(S) \geq t(S)|_e$. On the other hand we can derive a lower bound for $t(S)|_e$ by multiplying the number of triangulations for the two sets S_1 and S_2, i.e., $t(S)|_e \geq t(S_1) \cdot t(S_2)$. Since for any set S of cardinality $n \geq 4$ we can find an appropriate segment $e \in E(S)$ we get

Lemma 1 *$t(n) \geq t(n_1) \cdot t(n_2)$ for $n_1 + n_2 = n+2$, $n \geq 4$, $n_1, n_2 \geq 3$.*

A solution of the recurrence relation of Lemma 1 is given by

$$t(n) \geq \frac{1}{\tau^2} \cdot \tau^n \qquad (1)$$

for any constant $\tau > 0$. This yields an asymptotic bound of $t(n) = \Omega(\tau^n)$. To get the recurrence started,

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[1] All point sets considered in this paper are assumed to be in general position, i.e., no three points are collinear. Note that the general position assumption is crucial to avoid trivialities.

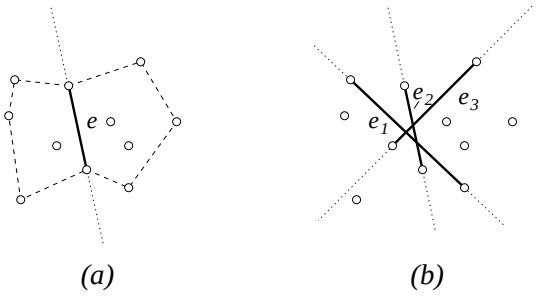

Figure 1: (a) Segment e splitting the point set into subsets of cardinalities 6 and 7. (b) Three pairwise crossing segments splitting 6:7, 6:7 and 5:8.

values of τ for instances of small constant size have to be determined. Let us point out that (1) is a solution for *any* base τ, thus values of τ as large as possible are sought for small point sets. In fact, choosing an appropriate *induction base* for Lemma 1 with 'good values' of τ will lead to a 'high' constant for (1). In other words, the quality of the initial values affects the asymptotic behavior of $t(n)$.

The above inequalities are rather loose, since only triangulations including a certain edge are considered. A generalization of Lemma 1 can be achieved by the following observation. Let $e_1, e_2 \in E(S)$ be two pairwise crossing segments spanned by S, cf. Figure 1 (b). Let $T(S)|_{e_1}$ and $T(S)|_{e_2}$ be the sets of triangulations of S including e_1 and e_2, respectively. Then $T(S)|_{e_1} \cap T(S)|_{e_2} = \emptyset$, i.e. no triangulation of S belongs to both sets. This follows from the simple fact that triangulations contain no crossing edges.

Thus if a point set admits $k \geq 2$ pairwise crossing segments we can apply the recurrence of Lemma 1 k times, leading to

Corollary 1 *$t(n) \geq \sum_{i=1}^{k} t(n_{1_i}) \cdot t(n_{2_i})$ for $n_{1_i} + n_{2_i} = n+2$, $n \geq 2k$, $n_{1_i}, n_{2_i} \geq k+1$, $1 \leq i \leq k$.*

Note that the lower bound on n_{1_i}, n_{2_i} stems from the fact that any involved segment is crossed by at least $k-1$ other segments, giving $k-1$ points on each side of its supporting line.

Solving the above recurrence relation gives

$$t(n) \geq \frac{1}{k\tau^2} \cdot \tau^n \tag{2}$$

where $\tau > 0$ is constant. For the asymptotic bound this still gives $t(n) = \Omega(\tau^n)$. But for the important task of determining τ for small instances this leads to a significant improvement (details are given in the next section and in Table 1). If for some small constant n we can show that any set S of size n exhibits $t(n)$ triangulations, then (2) provides a lower bound on τ for this fixed cardinality, denoted by $\tau(n)$:

$$\tau(n) \geq (k \cdot t(n))^{\frac{1}{n-2}}. \tag{3}$$

Once the range of the induction base is fixed, say $n_1 \leq n \leq n_2$ (see next sections for details) we get

$$\tau = \min_{n_1 \leq n \leq n_2} \tau(n) \tag{4}$$

giving the constants in our asymptotic lower bound (2). This again reveals the importance of the induction base. Note that k pairwise crossing segments are only needed for sets of cardinality *larger than* the induction base, not for instances of the induction base itself.

3 Induction Base

Let us again stress the fact that we are here in the remarkable situation that any improvement for small sets will immediately yield an improvement in the asymptotics of $t(n)$. Therefore the present and next sections are dedicated to derive a good induction base.

For $n \leq 10$, the minimum number of triangulations can be determined exactly by exhaustive counting them for each possible order type. A data base for all order types of point sets in the plane has recently been developed, see [2] for details. The second column of Table 1 is taken from [3] and gives exact lower bounds on the number of triangulations of n points, i.e. the value of $t(n)$, for $n = 3, ..., 10$.

Table 1 reflects the known fact that n points in convex position, whose number of triangulations is given by the Catalan numbers C_{n-2}, do not lead to the minimum. This is in contrast to other structures like e.g. crossing-free matchings and crossing-free spanning trees, where this happens to be true [7]. We conjecture that the minimum is always obtained by a special structure with only $\lfloor n/2 \rfloor$ extreme points, the so-called double circle, cf. Figure 2 (a), which can be shown to have $(\sqrt{12})^{n-\Theta(\log(n))}$ triangulations [8]. The conjecture is known to be true for $n \leq 10$ [3].

n	$t(n)$	$k=1$	$k=2$	$k=3$	$k=4$
3	1	1.00000	2.00000	3.00000	4.00000
4	1	1.00000	1.41421	1.73205	2.00000
5	2	1.25992	1.58740	1.81712	2.00000
6	4	1.41421	1.68179	1.86121	2.00000
7	11	1.61539	1.85560	2.01235	2.13153
8	30	1.76273	1.97860	2.11693	2.22091
9	89	1.89882	2.09647	2.22149	2.31469
10	250	1.99408	2.17456	2.28761	2.37137

Table 1: Values of $\tau(n)$ for small instances and crossing families of size 1 to 4.

A *crossing family* of size k (k-family for short) of a point set S is a set of k segments from $E(S)$ which pairwise cross each other. A useful result is taken from [3]:

Observation 1 *Every set S of $n \geq 10$ points admits a crossing family of size 3. The bound is tight in n.*

The best previous bound derives from an interesting result by Tóth and Valtr [10] on convex sets: among any $\binom{2c-5}{c-2}+2$ points there are at least c points in convex position, providing a crossing family of size $\lfloor \frac{c}{2} \rfloor$. This evaluates to $n \geq 37$ for $k=3$. In [4] the existence of crossing families of size $\sqrt{n/12}$ for every set of n points is proven. For $k=3$ this gives the weaker bound $n \geq 108$ but for larger k it is superior to the bounds in [10].

As mentioned in Section 2 we can utilize a crossing family of size k to increase values of $\tau(n)$. Table 1 shows the values of $\tau(n)$ for $k=1,...,4$. Observation 1 together with (2) and (4) yields $k=3$, $\tau = 1.73205$ and $t(n) \geq 0.11 \cdot 1.73^n$, a first substantial improvement over previous lower bounds.

From Table 1 it can be seen that values of τ are especially small for instances of size $n=4$ to 6. Since we aim at $\tau = 2+\varepsilon$ we have to avoid too small subsets in the recurrence, i.e. we have to split a set in a way that both sides get at least some fixed number of points. To this end let for a set S of size n an *ℓ-segment*, $2 \leq \ell \leq (n+2)/2$, be a segment $e \in E(S)$ such that the supporting line of e splits S into subsets of cardinality n_1 and n_2 with $\min\{n_1, n_2\} \geq \ell$. For example the convex hull of S consists of 2-segments. Similarly define an *$\hat{\ell}$-segment* with $\min\{n_1, n_2\} = \ell$.

Taking ℓ-segments for the recurrence of Corollary 1 allows us to start with the induction base at sets with cardinality ℓ. While to guarantee the existence of a single ℓ-segment it is sufficient to have $n = 2\ell-2$ points, the situation for k-families consisting of ℓ-segments is more involved. For example the case we are interested in are 3-families of 8-segments (Table 1 reveals some reasons for this choice). An obvious but useful result is

Observation 2 *A set of $2\ell-2$ points in convex position contains a crossing family of size $\ell-1$ entirely consisting of ℓ-segments.*

For $\ell = 8$ we get $n \geq 1352080$ from the above mentioned result on convex sets [10]. This can be improved to $n \geq 588$ by the results of [4] noting that any $\ell+1$-family consists of ℓ-segments. To avoid too large induction bases we improve on these results for 3-families in the next lemma.

Lemma 2 *Let $c(k)$ be the smallest number such that for any set of size $c(k)$ a crossing family of size k exists. Then for any set S of size $n \geq c(k) + 2\ell^2 - 2\ell k - 5\ell + 3k + 3$ there exists a crossing family of size k entirely consisting of ℓ-segments.*

Proof. From Observation 2 it follows that any convex subset of S of size $2\ell-2$ would be sufficient. So assume that all convex subsets have size at most $2\ell-3$. Consider the $\ell-k-1$ outermost convex layers of S, consisting of at most $(2\ell-3)(\ell-k-1) = 2\ell^2-2\ell k-5\ell+3k+3$ points. The remaining set S_c of at least $c(k)$ points exhibits a crossing family of size k, each segment having at least $k+1$ points of S_c on each side of its supporting line. In addition each of the $\ell-k-1$ convex layers (note that they are 'around' S_c) adds at least one point on both sides of an segment, thus all segments are $(k+1)+(\ell-k-1) = \ell$-segments. □

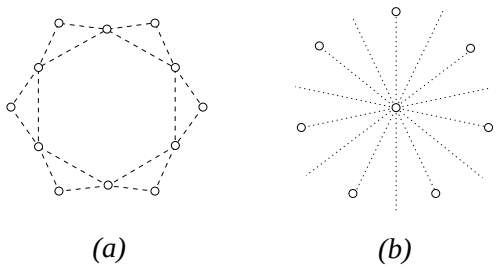

Figure 2: (a) Double circle, conjectured to minimize the number of triangulations; $\hat{3}$-segments are dashed (b) A point set without crossing halving segments.

Together with Observation 1 we get

Corollary 2 *For any set S of size $n \geq 22 + 2\ell^2 - 11\ell$ there exists a crossing family of size 3 entirely consisting of ℓ-segments.*

For large enough crossing families and ℓ-segments of suitable size, Lemma 2 and Corollary 2 give lower bounds for the range where the recurrence relation in Corollary 1 can be applied. In other words, if our induction base starts at ℓ we have to increase its upper boundary accordingly. E.g. for 8-segments and 3-families we get a range of $n = 8,...,62$. Therefore the next section is entirely devoted to provide an extended induction base.

4 Extended Induction Base

In the following we give various relations to extend the induction base beyond $n=10$. Let us point out that none of these relations leads to an direct improvement of the asymptotics of $t(n)$ but are rather utilized to compute concrete values of $t(n)$ for small, constant values of n.

From Lemma 1 we get

$$t(n) \geq \max_{3 \leq i \leq n-1} \{t(i) \cdot t(n+2-i)\} \text{ for } n \geq 4 \qquad (5)$$

For $n \geq 10$ there always exist three crossing segments (cf. Observation 1) but in general we do not have the freedom to choose the segments, i.e. we have to consider the 'worst' splitting for each segment, yielding

$$t(n) \geq \min_{4 \leq i \leq n-2} \{3 \cdot t(i) \cdot t(n+2-i)\} \text{ for } n \geq 10 \quad (6)$$

For $n \geq 5$ we have a crossing family of size 2, and for $n \geq 6$ we can always find one consisting of either

two $\hat{3}$-segments or of one $\hat{3}$- and one $\hat{4}$-segment. Since $t(n) \geq t(n-1)$ this leads to a Fibonacci-like relation.

$$t(n) \geq t(n-1) + t(n-2) \text{ for } n \geq 6 \qquad (7)$$

Proof. Omitted in this extended abstract.

Note that from the existence of two pairwise crossing $\hat{3}$-segments we could derive $t(n) \geq 2 \cdot t(n-1)$ and thus $t(n) = \Omega(2^n)$. However, such structures need not always exist. Interestingly the mentioned double circle already constitutes a counter example, cf. Fig. 2 (a).

On the other hand one could try to proof the existence of two crossing halving segments (in our case meaning $\ell = \lfloor n/2 \rfloor + 1$) to obtain $t(n) \geq 2 \cdot t(\lfloor n/2 \rfloor + 1)^2$, but Figure 2 (b) gives an example without such segments. The best we can prove is the existence of two crossing $(\lceil \frac{n}{2} \rceil - 2)$-segments, giving

$$t(n) \geq \min_{\lceil \frac{n}{2} \rceil - 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 4} \{2 \cdot t(i) \cdot t(n+2-i)\} \text{ for } n \geq 8 \qquad (8)$$

Proof. Omitted in this extended abstract.

n	$t(n)$	eqn	$k=1$	$k=2$	$k=3$	$k=4$
11	339	(7)	1.910	2.063	2.158	2.229
12	589	(7)	1.892	2.028	2.112	2.174
13	990	(6)	1.872	1.994	2.069	2.124
14	1 767	(6)	1.865	1.976	2.043	2.093
15	2 970	(6)	1.850	1.951	2.013	2.058
16	7 921	(5)	1.899	1.995	2.054	2.096
17	22 250	(5)	1.949	2.041	2.097	2.138
18	62 500	(5)	1.994	2.082	2.136	2.175
19	84 750	(5),(7)	1.949	2.030	2.079	2.115
20	147 250	(5),(7)	1.937	2.013	2.059	2.092
21	314 526	(8)	1.947	2.019	2.063	2.094
22	528 660	(8)	1.933	2.001	2.042	2.071
⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	$1.4 \cdot 10^{17}$	(5),(7)	1.976	2.000	2.014	2.024
61	$3.6 \cdot 10^{17}$	(5)	1.985	2.008	2.022	2.032
62	$6.7 \cdot 10^{17}$	(5)	1.982	2.005	2.019	2.029
min			1.850	1.951	2.013	2.024

Table 2: Extended induction base: $t(n)$, $\tau(n)$.

Formulas (5) to (8) are used to compute lower bounds on the number of triangulations for small, constant values of n. For any n the maximum among the obtained values is taken. The results are shown in Table 2 together with the number of the equations used to derive the bound, and values of $\tau(n)$ for families of size $k = 1, ..., 4$ (the full table is given in the full paper).

It can be seen that an induction base of range $8, ..., 62$ and using 3-families provides $\tau \approx 2.013$. Together with (2) and Corollary 2 this proves Theorem 1.

5 Discussion

Using the results of Sections 3 and 4 the extended induction base might be further expanded, e.g. up to cardinalities of several thousand. On the other hand crossing families of size 4 or more might be considered, again leading to a larger induction base. Both would significantly increase the complexity of the approach, e.g. for a 4-family consisting of 7-segments the induction base would range from 7 to 214. Unlike suggested by the results of Table 2 for $k = 4$ computer experiments showed no improvements.

Most promising attacks to improve on τ are to determine the exact lower bounds for sets of size $n = 11, 12, ...$ or to show the existence of k-families for $k \geq 4$ consisting of, say, 10-segments for relatively small point sets, in order to avoid huge induction base ranges.

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