Guarding Lines and 2-Link Polygons is APX-Hard

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Abstract

We prove that the *minimum line covering problem* and the *minimum guard covering problem* restricted to 2-link polygons are APX-hard.

keywords: Computational Geometry, Polygon Decomposition, Art Gallery Theorem, Minimum Guard Covering, Minimum Line Covering.

1 Introduction

Picture yourself as the president of the national museum in your country holding invaluable treasures of history and art. How many surveillance cameras would you need installed at the museum to make you sleep tight at night? Being a scientist you might come to the conclusion that if every part of the gallery is seen by the cameras, this would fulfill your needs. The answer you seek is the solution to the *Art Gallery Problem* [4, 9, 10]: "how many cameras do we need to guard a given gallery and how do we decide where to place them?" This problem was posed by Viktor Klee in 1973 in response to a question raised by Vašek Chvátal [6]. The latter showed in 1975 that $\lfloor n/3 \rfloor$ cameras are sufficient and sometimes necessary to guard any gallery represented as a two dimensional simple polygon [3].

The algorithmic version of this problem is to find a minimum number of guards for a given art gallery. This is called the *minimum guard covering problem* and was shown to be NP-hard by Aggarwal [1] and Lee and Lin [8]. This is actually a polygon decomposition problem. The visibility polygon from a point, that is the area a given guard can see, is star shaped. This makes the minimum guard coverage problem identical to that of finding a minimum star cover of a polygon. The problem is known to be computationally difficult [5] and it is strongly related to the minimum set cover problem.

In 1995 Joseph S. B. Mitchell raised the question of correspondence between link diameter and guardability of simple polygons. 1-link polygons are convex and hence trivially guardable in polynomial time. The original NP-hardness proofs [1, 8] immediately give NP-hardness for polygons with link diameter ≥ 4. Nilsson proved the NP-hardness for ≥ 3-link polygons [9]. This leaves the complexity question for the minimum guard covering problem for 2-link polygons open. Note that a large class of 2-link polygons are star shaped and therefore guardable in polynomial time. In this paper we set out to answer the complexity question for 2-link polygons. In doing so we analyze a related problem called the minimum line covering problem which is interesting in its own right. We give a succinct reduction from MAX 2SAT proving that this problem is APX-hard. A similar result was claimed by Kumar et. al. [7] but our construction gives an explicit lower bound on the approximation ratio.

From the minimum line covering problem there is a straightforward gap preserving reduction to the minimum guard covering problem, a fact that Mitchell and Konečný independently observed.

2 Problem Formulation

Before we prove our claim we give formal definitions of the Minimum Guard Covering Problem, the Minimum Line Covering Problem, and k-link polygons.

Minimum Guard Covering Problem (MGCP)
Instance: A polygon P.
Solution: A minimum set of points in P from which the entire polygon, interior and boundary can be *seen*. Two points $x, y \in P$ *see* each other if the straight line segment $\overline{xy} \subset P$.

The Minimum Line Covering Problem (MLCP)
Instance: A set L of non-parallel lines in the plane.
Solution: A minimum set P of points such that there is at least one point in P on each line in L.

Definition *A polygon P has* link diameter k *if k is the minimum integer value such that for all points $x, y \in P$ there exists a path from x to y that does not cross the boundary of P and that consists of k straight line segments. We say that a polygon is k link if it has link diameter k.*

The Minimum Guard Covering Problem is trivial for 1-link polygons since these polygons are convex. The original NP-hardness proofs of the Minimum Guard

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Covering Problem constructs a polygon instance having link diameter 4 [1, 8] and this was strengthened to 3-link polygons by Nilsson [9]. In the next section we prove that the Minimum Line Covering Problem is APX-hard. This, in turn, leads us to the APX-hardness result for the Minimum Guard Covering Problem in 2-link polygon instances, strengthening the previous results.

3 The Reduction

The reduction is made from a special case of MAX SAT where each clause contains two literals and each literal occurs at most twice.

MAX 2SAT(2L)
Instance: A set U of variables and a collection C of disjunctive clauses with exactly 2 literals. Each literal can appear at most twice in C. A literal is a variable or a negated variable in U.
Solution: A truth assignment for U that satisfies as many clauses as possible in C.

This problem is APX-hard, a direct consequence of the following lemma:

Lemma 3.0.1 (Berman & Karpinski [2]) *For every $\epsilon > 0$, it is hard to approximate 3-OCC-MAX 2SAT within factor $2012/2011 - \epsilon$.*

Here 3-OCC-MAX 2SAT is the MAX 2SAT problem where each variable can occur at most three times, a special case of MAX 2SAT(2L).

The reduction is divided into two parts. The first part reduces an arbitrary MAX 2SAT(2L) instance to the minimum line covering problem (MLCP). The second part reduces the MLCP to the minimum guard covering problem for 2-link polygon instances. Note that the instances covered by the MLCP all consists of non-parallel lines. This is important in the second part of the reduction, since we can construct a 2-link polygon from such an arrangement.

3.1 The Reduction to MLCP

Let (U, C) be a MAX 2SAT(2L) instance. We start the reduction by creating, for each variable $x_i \in U$, a set L_i of eight lines with intersections as in Fig. 1. Each set L_i forms a *consistency gadget* and they are built so that no two lines are parallel. Furthermore, there are no intersections between three lines except those explicitly described. Note that the lines in Fig. 1 that appear to be parallel really are not. In each L_i the lines a and c represent the literal x_i, and the lines b, d the literal $\neg x_i$. We call these lines *literal lines*.

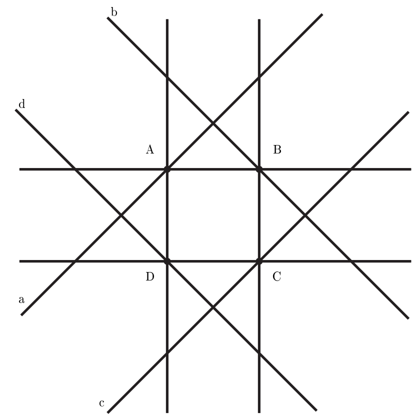

Figure 1: The consistency gadget.

For each clause $(l_i, l_j) \in C$ we create a line, denoted *clause line* that intersects all other lines in the construction.

Furthermore, for each literal in each clause we create an additional line. This line passes through the intersection point between the clause line and the literal line of the consistency gadget representing the corresponding variable; see Fig. 2. Note that there are two possible literal lines to choose from and that a literal in the MAX 2SAT(2L) instance can occur in at most two clauses. Thus, there is always a free literal line to choose from at any time in the construction. Note that the construction contains intersections of two and three lines only, and that no lines are parallel. The construction contains a total of $8|U|$ lines in consistency gadgets and $3|C|$ additional lines for the clauses, for a total of $3|C| + 8|U|$ lines. Let L be the set of lines thus constructed.

An *interesting set* S for L is a maximal set of points such that every point $x \in S$ is the intersection of three lines in L and every line in L contains at most one point from S. Note that on every clause line there are two intersection points that could be included in an interesting set and in every consistency gadget there are four intersection points. No other points can be included.

Lemma 3.1.1 *The size of a solution to the MLCP construction is*

$$\left\lceil \frac{3|C| + 8|U| - |S|}{2} \right\rceil,$$

where S is the maximum interesting set in the solution.

Proof: We can view the solution to the MLCP as follows. Every point in the maximum interesting set covers three lines. The remaining points in the solution

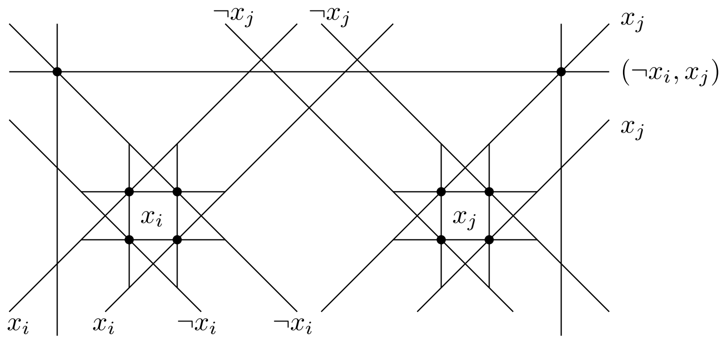

Figure 2: Where the lines from the literals in a clause cross the line for that clause we add a new line.

can cover at most two lines each. The total number of lines in the construction is $3|C|+8|U|$ so the solution's size is

$$\left\lceil \frac{3|C|+8|U|-3|S|}{2} \right\rceil + |S| \quad = \quad \left\lceil \frac{3|C|+8|U|-|S|}{2} \right\rceil .$$

□

We use the interesting set to describe a truth assignment to the corresponding MAX 2SAT(2L) instance and to count the number of clauses satisfied by the assignment. The interesting set restricted to a consistency gadget has a size no larger than two; see Fig. 1. These maximal sets are $\{A, C\}$ and $\{B, D\}$. When they occur in an interesting set they represent the value false ($\{A, C\}$) or true ($\{B, D\}$) of the corresponding variable. We say that the consistency gadget in this case is *set* either to false or true (by the interesting set). It is important to notice that three-way intersection points in different consistency gadgets are independent of each other in the sense that all consistency gadgets can be set arbitrarily to either true or false without violating the definition of interesting sets. However, they will put restrictions on the rest of the points in the interesting set. These points all lie on clause lines and we use them to count the number of clauses satisfied by the truth assignment. Let us assume that the interesting set contains a description of a truth assignment, that is, every consistency gadget is set to some value. By the definition of interesting set there can be at most one point in S on each clause line. This point represents a literal that satisfies the clause. It can be included in the interesting set if and only if the corresponding consistency gadget is set to a value that makes the literal satisfy the clause; see Fig. 3. This implies that the truth assignment described by the interesting set satisfies c clauses of the instance if and only if the set contains c points from the clause lines, i.e. the size of the interesting set is $2|U|+c$, still under the assumption that every consistency gadget is set to some value.

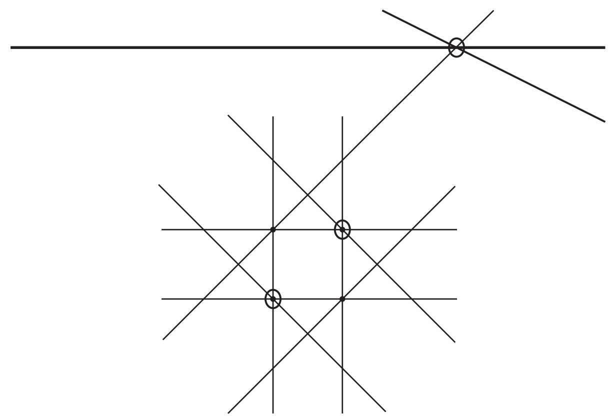

Figure 3: The intersection point on the clause line can be included in the interesting set if and only if the consistency gadget is set (to true in this example).

To complete the reduction we need to prove that there is always a maximum interesting set consistent with a truth assignment of the variables. If this is true then we have a direct correspondence between the size of the cover and the number of satisfied clauses. Assume that there is no maximum interesting set that corresponds to a truth assignment. Consider the maximum interesting set, S, that contains the largest number of points from the consistency gadgets. There is at least one consistency gadget G that is not set, i.e., it contains less than two points in S. Set the gadget G by fixing one or two points appropriately. Let $p \in G$ be one of these points. The reason for p not being in S is that the literal line that contains p also contains a point $q \in S$. If we remove q from S then we can include p in S, thereby getting a new interesting set S' with $|S| = |S'|$ but with one more point from the consistency gadgets, contradicting our choice of S and hence our assumption that there is no maximum interesting set corresponding to a truth assignment.

The thorough reader notices that there is one more thing to prove before we can conclude that the reduction can be done in polynomial time, i.e., that the description of the lines is polynomial in the size of the 3-OCC-MAX 2SAT instance. To see this we give a description of how the lines are placed in the plane. First the consistency gadgets are placed evenly spaced as in Figure 2. Next we rotate the first gadget $45/|U|$ degrees, the second one twice as much, the third one three times as much and so forth. Since the angle between two lines in the consistency gadget is no less than 45 degrees this means that all lines intersect. We place the $|C|$ clause lines in such a way that no two lines are parallel. This can be done easily. Now, we place the extra lines at the intersection points between the clause lines and the relevant literal lines. This clearly gives a polynomial description of the lines. We have proved the following theorem.

Theorem 3.1.2 *For every $\delta > 0$, it is hard to approximate MLCP within factor $28169/28168 + \delta$.*

PROOF: Given an instance to MAX 2SAT(2L), let c denote the maximum number of clauses satisfiable. We can bound the number of variables to $|U| \leq 2|C|$, since each clause contains 2 literals. From Lemmas 3.0.1, 3.1.1, and the following discussion we infer that the inapproximability ratio is

$$\frac{\left\lceil \frac{3|C|+6|U|-2011c/2012+\epsilon}{2} \right\rceil}{\left\lceil \frac{3|C|+6|U|-c}{2} \right\rceil} \quad \geq \quad 28169/28168 + \delta,$$

since the worst case occurs when $c = |C|$. The δ depends on ϵ.

$\square$

3.2 The Reduction to MGCP

In the second part of the reduction we are given an arrangement of non-parallel lines. A rectangle R sufficiently large to contain all intersections between lines in L is created. Consider a line segment in $R \cap L$. At one end point we put a spike in the rectangle; see Fig 4. The cones visible from any triplet of spikes should intersect if and only if the corresponding line segments cross. If this criterion is met then the intersections between line segments will correspond to areas in this polygon. Thus, from a guard set in the polygon we get a point set in the MLCP. That is, if we can solve the minimum guard covering problem then we immediately get a solution to the MLCP.

Theorem 3.2.1 *The minimum guard covering problem is APX-hard.*

4 Conclusion

We have proved that the minimum guard covering problem restricted to 2-link polygons is APX-hard. In addition we give an explicit lower bound on the approximation ratio of the minimum line covering problem.

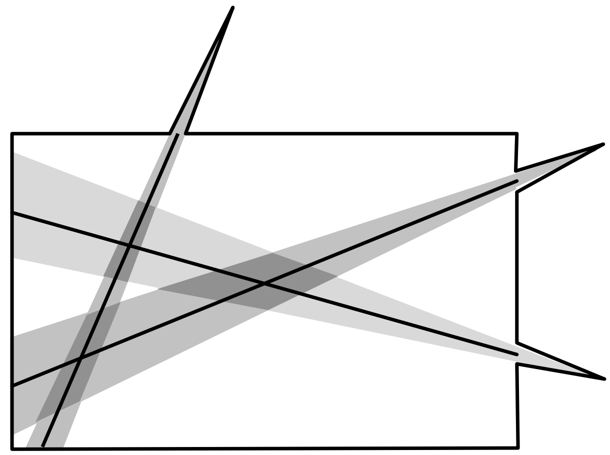

Figure 4: The set of lines are embedded inside a rectangle. Where the resulting line segments meet the boundary we create a narrow spike.

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