Good Approximations for the Relative Neighbourhood Graph

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Abstract

The Urquhart graph of a set of points in the plane is obtained by removing the longest edge from each triangle in the Delaunay triangulation. We show experimental evidence that the Urquhart graph is a good approximation for the relative neighbourhood graph in the sense that it contains few additional edges. For random samples, the Urquhart graph is typically only about 2% larger than the relative neighbourhood graph, and thus may serve equally well for computational morphology tasks.

1 Introduction

The *relative neighbourhood graph* $\mathrm{RNG}(S)$ of a set S of points in the plane is defined as follows: Two points p and q in S define an edge of $\mathrm{RNG}(S)$ when

$$d(p,q) \leq \max\{d(p,r), d(q,r)\}$$

for all points r in S, where d is the Euclidean distance. This graph was introduced by Toussaint [5] as tool for *computational morphology*, which is the computational extraction of perceptually meaningful structure from dot patterns. Like several other proximity graphs, $\mathrm{RNG}(S)$ is a subgraph of the Delaunay triangulation $\mathrm{DT}(S)$ of S.

If we relax the condition defining $\mathrm{RNG}(S)$ by requiring that it only holds for the points r that are adjacent to p and q in $\mathrm{DT}(S)$, then we obtain a supergraph $\mathrm{UG}(S)$ of $\mathrm{RNG}(S)$, called the *Urquhart graph* of S. Geometrically, $\mathrm{UG}(S)$ is obtained from $\mathrm{DT}(S)$ by removing the longest edge from each triangle in $\mathrm{DT}(S)$. Urquhart [6] proposed this very method for computing $\mathrm{RNG}(S)$ efficiently, but Toussaint [4] gave a counter-example showing that in general $\mathrm{UG}(S)$ is strictly larger than $\mathrm{RNG}(S)$.

In this note, we show experimental evidence that $\mathrm{UG}(S)$ is a good approximation for $\mathrm{RNG}(S)$, in the sense that it contains few additional edges. Since $\mathrm{UG}(S)$ is easily computed from $\mathrm{DT}(S)$, and there are several excellent and fast implementations of $\mathrm{DT}(S)$, it is likely that $\mathrm{UG}(S)$ may serve equally well for computational morphology tasks that currently use $\mathrm{RNG}(S)$.

It is easy to reduce the cubic brute-force algorithm for computing $\mathrm{RNG}(S)$ to a quadratic algorithm by testing only Delaunay edges instead of all pairs of points. Supowit [3] gave the first optimal algorithm for computing $\mathrm{RNG}(S)$: it extracts $\mathrm{RNG}(S)$ from $\mathrm{DT}(S)$ in time $O(n \log n)$. Lingas [2] gave a simple algorithm that extracts $\mathrm{RNG}(S)$ from $\mathrm{DT}(S)$ in time $O(n)$, but it has not been implemented (personal communication).

On the other hand, the Urquhart graph is easy to compute. Since $\mathrm{DT}(S)$ has $O(n)$ edges, it is clear that $\mathrm{UG}(S)$ can be extracted from $\mathrm{DT}(S)$ time $O(n)$. Since $\mathrm{DT}(S)$ can be computed in time $O(n \log n)$, this gives a simple algorithm for $\mathrm{UG}(S)$ with total time $O(n \log n)$. Thus, $\mathrm{UG}(S)$ is a good tool for computational morphology tasks, because it approximates $\mathrm{RNG}(S)$ well — as the experiments reported below indicate — and can be extracted easily from $\mathrm{DT}(S)$.

2 The experiments

To test how well the Urquhart graph $\mathrm{UG}(S)$ approximates the relative neighbourhood graph $\mathrm{RNG}(S)$, we selected random samples from several regions in the plane and compared the number of edges in $\mathrm{RNG}(S)$, $\mathrm{UG}(S)$, $\mathrm{GG}(S)$, and $\mathrm{DT}(S)$.

We included the *Gabriel graph* $\mathrm{GG}(S)$, whose edges correspond to pairs of points p and q of S such that the circle having diameter pq contains no other point of S, because $\mathrm{GG}(S)$ is a well-known proximity graph that lies between $\mathrm{RNG}(S)$ and $\mathrm{DT}(S)$. In fact, we have the following inclusion relations [4]:

$$\mathrm{RNG}(S) \subseteq \mathrm{UG}(S) \subseteq \mathrm{GG}(S) \subseteq \mathrm{DT}(S).$$

Figures 1–4 show these graphs for a random sample of 1000 points in the unit square, on a spiral, and on the boundary of two line-art pictures. Note that $\mathrm{UG}(S)$ is very similar to $\mathrm{RNG}(S)$, whereas $\mathrm{GG}(S)$ is clearly different from both $\mathrm{RNG}(S)$ and $\mathrm{UG}(S)$. In Figures 1 and 2 it is not easy to spot the additional edges in $\mathrm{UG}(S)$ (22 for the square, 10 for the spiral). In Figures 3 and 4 it is easier to spot some of the additional edges in $\mathrm{UG}(S)$ — perhaps because of the visual content of the pictures — but the additional edges are still few (27 for the earth, 19 for the man).

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Table 1: Random points in the unit square.

	100 points		500 points		1000 points		2000 points		5000 points	
	edges	%	edges	%	edges	%	edges	%	edges	%
RNG	119.2	1.000	621.4	1.000	1250.6	1.000	2511.6	1.000	6310.4	1.000
UG	120.8	1.013	631.6	1.016	1275.8	1.020	2558.4	1.019	6432.6	1.019
GG	179.6	1.507	947.2	1.524	1943.4	1.554	3902.0	1.554	9833.8	1.558
DT	285.6	2.396	1479.6	2.381	2977.2	2.381	5977.2	2.380	14971.6	2.373

Table 2: Random points in the unit disk.

	100 points		500 points		1000 points		2000 points		5000 points	
	edges	%	edges	%	edges	%	edges	%	edges	%
RNG	119.6	1.000	621.4	1.000	1262.4	1.000	2520.4	1.000	6333.4	1.000
UG	122.6	1.025	632.2	1.017	1284.4	1.017	2571.6	1.020	6452.2	1.019
GG	188.4	1.575	961.4	1.547	1948.6	1.544	3937.4	1.562	9837.4	1.553
DT	283.4	2.370	1476.4	2.376	2971.6	2.354	5953.0	2.362	14939.2	2.359

Table 3: Random points on a spiral.

	100 points		500 points		1000 points		2000 points		5000 points	
	edges	%	edges	%	edges	%	edges	%	edges	%
RNG	128.2	1.000	658.0	1.000	1304.8	1.000	2579.2	1.000	6333.2	1.000
UG	129.4	1.009	661.4	1.005	1313.2	1.006	2597.2	1.007	6388.8	1.009
GG	174.4	1.360	848.6	1.290	1669.6	1.280	3245.6	1.258	7951.8	1.256
DT	270.8	2.112	1368.8	2.080	2750.8	2.108	5498.4	2.132	13772.8	2.175

Tables 1–3 show the average results for several random samples in the unit square, in the unit disk, and on a spiral. For each graph and sample size, we give the average number of edges in the graph and its relative size with respect to the relative neighbourhood graph.

3 Conclusion

The empirical evidence indicates that the Urquhart graph UG(S) approximates the relative neighbourhood graph RNG(S) well, being typically only about 2% larger than the RNG(S) for random samples. Thus, UG(S) may serve equally well for computational morphology tasks, while being easy to compute from the Delaunay triangulation DT(S).

It would be interesting to have probabilistic results about the expected number of edges in UG(S) similar to those proved by Devroye [1] for RNG(S) and GG(S).

It would also be interesting to compare actual implementations of the algorithms by Supowit [3] and Lingas [2] with the extraction of UG(S) from DT(S).

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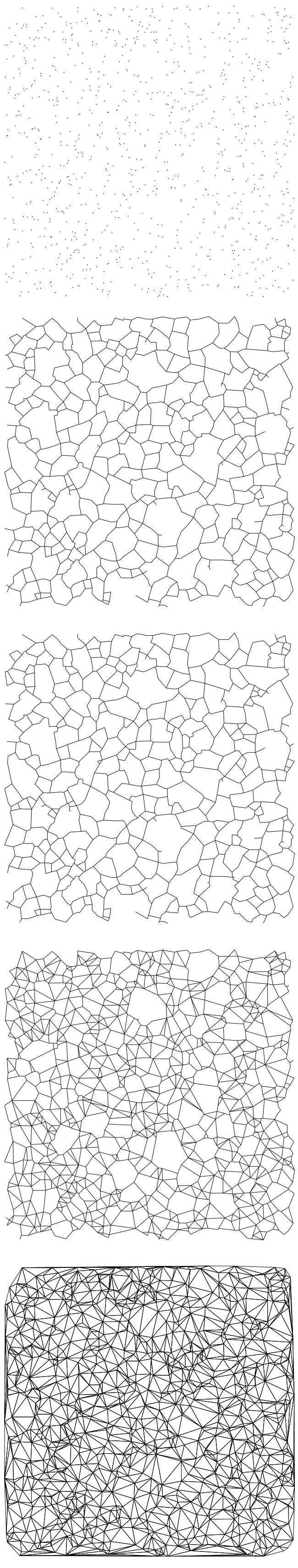


S

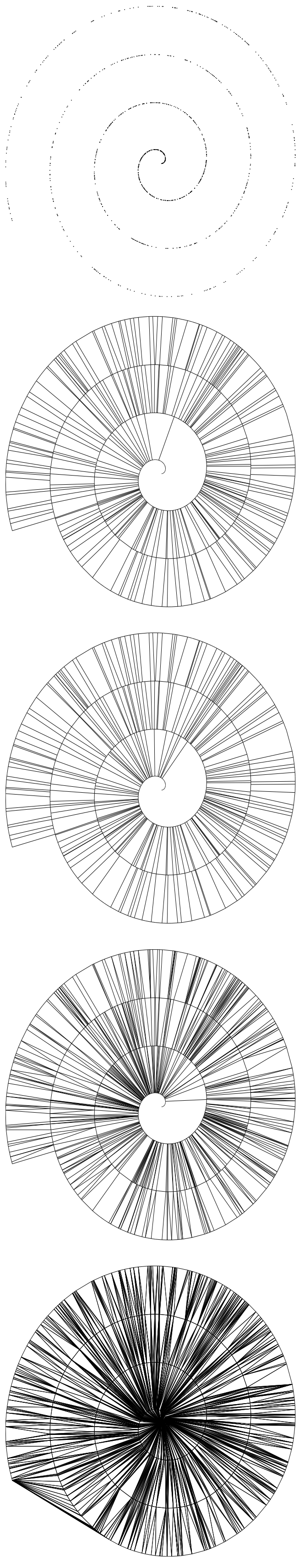
$\mathrm{RNG}(S)$

$\mathrm{UG}(S)$

$\mathrm{GG}(S)$

$\mathrm{DT}(S)$

Figure 1: The graphs for a random sample of 1000 points in the unit square.



S

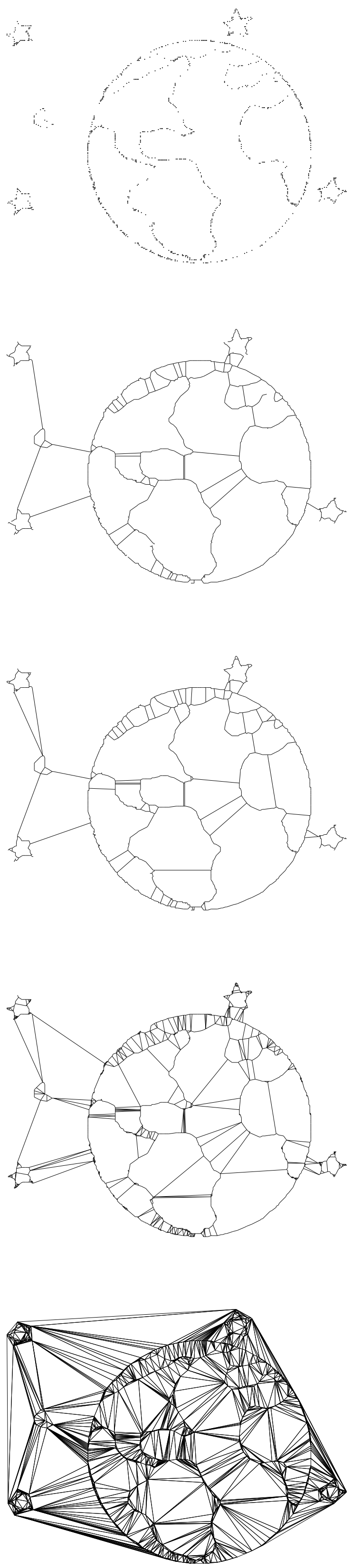
$\mathrm{RNG}(S)$

$\mathrm{UG}(S)$

$\mathrm{GG}(S)$

$\mathrm{DT}(S)$

Figure 2: The graphs for a random sample of 1000 points on a spiral.



S

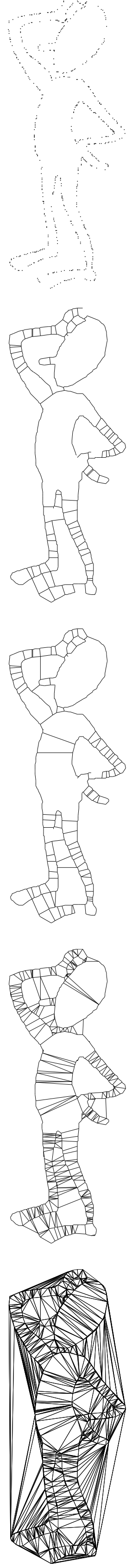
$\mathrm{RNG}(S)$

$\mathrm{UG}(S)$

$\mathrm{GG}(S)$

$\mathrm{DT}(S)$

Figure 3: The graphs for a random sample of 1034 points on a line-art picture.



S

$\mathrm{RNG}(S)$

$\mathrm{UG}(S)$

$\mathrm{GG}(S)$

$\mathrm{DT}(S)$

Figure 4: The graphs for a random sample of 597 points on a line-art picture.