Local Optimization of Triangular Surface Meshes for General Quadrics in L_p Norm

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Abstract: For a set S of distinct points in the *x-y* plane and a 3-dimensional surface with the general quadratic equation $z = ax^2 + by^2 + cxy + dx + ey + f$ we study the problem of finding a locally optimal triangulation of S for the linear approximation of the surface under the L_p norm. We show that the L_p norm error is independent of the translation and rotation by 180 degrees of a single triangle in the *x-y* plane. This observation allows us to substantially simply the local optimization procedure, and in particular allows us to explicitly compute the separation curves. Locally optimal triangulations are important in the data dependent approximation methods. The paper generalizes earlier results by P. Desnogues and O. Devillers (presented at the Canadian Conference on Computational Geometry CCCG '95), who studied optimal local triangulations for the unit hyperbolic paraboloid in the L_2 norm error.

Introduction

A triangular surface mesh for a 3-D surface can be constructed starting with a set S of points in x-y plane and the corresponding data values on the surface. Any triangulation of S induces a linear approximation (triangular mesh) of the surface such that the projection of the mesh is this triangulation. In the area of data dependent triangulations the following definitions are common.

Let d be an interior edge of the triangulation T(S), and let Q be the quadrilateral formed by the two triangles sharing d. We say that d of T(S) is *locally optimal* with respect to a given cost function C if either Q is non-convex or Q is strictly convex and C(T(S)) < C(T'(S)) (or they are equal), where T'(S) is a triangulation obtained from T(S) by flipping d with the other diagonal of Q. Moreover, we say that T(S) is locally optimal with respect to C if all of its edges are locally optimal.

We can always construct a locally optimal triangulation by flipping diagonals until all of them are locally optimal. Such a procedure always terminates as each flipping operation reduces the cost and there is only a finite number of possible triangulations.

Finally, the *separation curve* of a triangle t with vertices in S determines the zone where no other vertices of triangles neighboring t may lie for t to be a triangle in the locally optimal triangulation of S. The circumcircle of a triangle is a well-know separation curve for the Delaunay triangulation.
This paper closely follows earlier results by P. Desnogues and O. Devillers [1], who provided simple rules for identifying non-optimal diagonals for the unit hyperbolic paraboloid under the L_2 norm error. We extend their results to the general quadrics (actually, they conjectured that the generalization would be possible) and to a more general L_p norm.

L_p norm and the error of linear approximation

Assume that we have a Δabc where the coordinates of a, b, c are: $a(x_0, y_0)$, $b(x_1, y_1)$, $c(x_2, y_2)$ in the $z=0$ plane in a given triangulation and the lifted images of these three points on the 3-dimensional surface are $A(x_0, y_0, z_0)$, $B(x_1, y_1, z_1)$, $C(x_2, y_2, z_2)$, respectively. If we draw a vertical line passing through a given point *p(x, y)* inside Δ*abc*, this line will pass through the surface and the triangle determined by A, B and C. Let us denote these two points by $p_1(x, y, u)$, $p_2(x, y, v)$, respectively. The error of the linear approximation of the quadric with ΔABC is defined as:

$$L_p(\Delta abc) = \iint_{\Delta abc} \left[p_1(x, y, u) - p_2(x, y, v) \right]^p dxdy \,. \quad (1)$$

$L_p(\Delta abc)$ is called the approximation error with norm L_p for Δ*abc*. Note that formally the L_p norm is defined as the *p*-th root of (1).

For any given four points that form a convex quadrilateral in the x-y plane, one may construct two different triangulation as shown in Figure 1.

A simple question about the approximation is: *Which one is better with respect to the above approximation error*?

The answer is in the difference

$$Diff(abcd) = L_2(\Delta abc) + L_2(\Delta bcd) - L_2(\Delta abd) - L_2(\Delta acd) \quad (2)$$

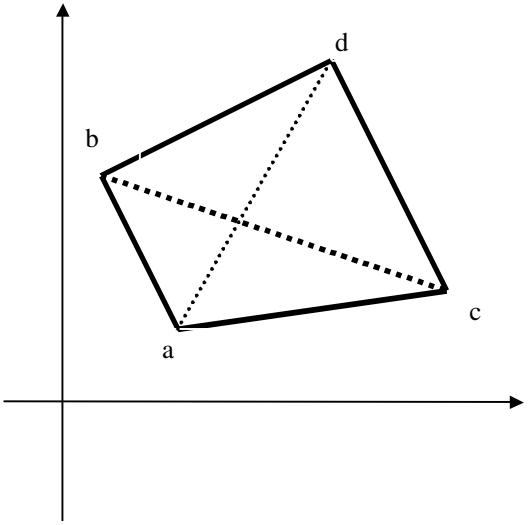

Figure 1: *Flipping diagonals change triangulations for four points that form a convex quadrilateral*

If *Diff(abcd)>0* then we use the lifted triangles *ABD* and *ACD* to approximate the original 3-dimensional surface; otherwise we use triangles *ABC* and *BCD*. For a given triangulation, if we use this criterion to test each pair of adjacent triangles until no common edge of any two of them needs to be swapped, then the final triangulation is locally optimal.

In addition, assume that the triangle *abc* is given and the point *d* is able to arbitrarily move in the x-y plane then the curves defined by *Diff(abcd)=0* divide the x-y plane into different regions (zones) that only depend on the coordinates of the points *a, b,* and *c*. The curves satisfying *Diff(abcd)=0* with respect to the coordinate of the point *d* are the separation curves of the triangle *abc* and they are independent of the fourth point *d*.

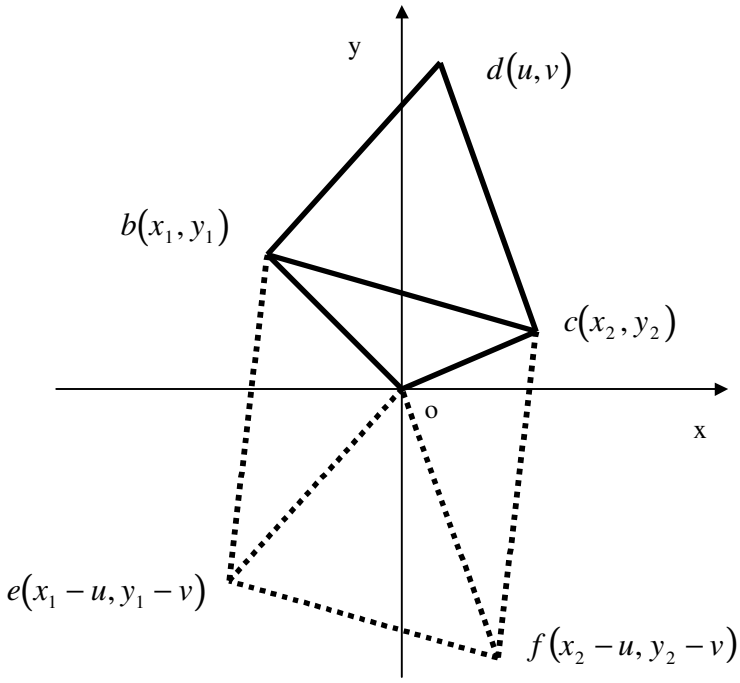

Figure 2: *Translation and mirror projection simplify calculations*

Main results

1. We prove that the integral given in Equation (1) is independent of the translation of Δabc and of its rotation by 180 degrees in the $z = 0$ plane.

2. Based on the above result, we can simplify the evaluation of the integral for any triangle in the $z = 0$ plane by just shifting one vertex of the triangle to the origin of the coordinate system.

This simple operation allows us to use alternative triangles as depicted in Figure 2. Instead of computing the error for triangles based on vertices *a* (translated to o), *b* and *c*, we use triangles *beo, efo, fco* and *cbo*; each of them has one vrtex in the origin of the coordinate system.

3. The simplification provided by moving one vertex to the origin allows us to use symbolic calculation tools (we use Maple as a tribute to Waterloo [2]) to calculate the integrals in Equation (2) and establish a simple expression for the difference *Diff(abcd)* (actually the difference is just a product of two quadratic equations - we derive such formulas for $p = 2$). Without the simplification the number of terms is prohibitive. The decision to swap a diagonal of two adjacent triangles can be determined efficiently by evaluating the sign of the expression.

The formula for $p = 2$ is

Diff(obcd)=g(u, v) h(u, v)* where

$$g(u,v) = cx_1y_1 + cx_2y_2 + by_2^2 + bv^2 + by_1^2 + ax_2^2 + ax_1^2 + au^2 - \frac{1}{3}cx_1y_2 - \frac{1}{3}cx_1v - \frac{1}{3}cx_2y_1$$

$$-\frac{1}{3}x_2v-\frac{2}{3}by_2v-\frac{2}{3}by_1v+cuv-\frac{1}{3}cy_2u-\frac{1}{3}cy_1u$$
$$-\frac{2}{3}by_1y_2-\frac{2}{3}ax_2u-\frac{2}{3}ax_1u-\frac{2}{3}ax_1x_2$$

(2)

and

$$h(u,v)=ax_1x_2v-by_1x_2v^2+by_1^2x_2v-ax_1x_2^2v$$
$$-ay_1x_2u^2+ay_1x_2^2u+by_1y_2^2u+cy_2x_1uv$$
$$+cy_1y_2x_2u+by_2x_1v^2-by_2^2x_1v-cy_1y_2x_1u$$
$$-cy_1x_2uv+cy_1x_1x_2v-cy_2x_1x_2v$$
$$-ay_2x_1^2u-by_1^2y_2u+ay_2x_1u^2 \qquad (3)$$

Based on the product *g(u, v)*h(u, v)*, the original integrals can be simplified and the decision to flip the diagonals of the adjacent triangles can be easily determined by substituting the corresponding coordinates of the vertices of the two adjacent triangles into the above product of the polynomials after translating the origin of the coordinate system to one vertex of the triangle.

4. Based on the product of two quadratic equations for *p = 2,* as presented above, we easily obtain the separation curves for the given triangle. The separation curves determined by *Diff(obcd)=0* can be simply represented by two set of curves: one is the curve *g(u, v)* as shown in Equation (2) and the other is the intersection of two 3-dimensional surfaces as shown in the following Equation (4).

$$(4)\quad \begin{cases} z=mu+nv+q \\ z=au^2+bv^2+cuv+du+ev+f \end{cases}$$

5. We implement the above method to display the results: the separation curves of a triangle, the initial triangulation (we use Delaunay triangulations [3] in our experiments) and the resulting locally optimized triangulation.

Proofs and details are presented in the full version of this paper.

References

[1] Desnogues, P. and Devilers. O., "*A Locally Optimized Triangulation of the Hyperbolic Paraboloid"*, Proceedings of the 7-th Canadian Conference on Computational Geometry, Quebec City, Quebec, Canada, 1995

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[3] Aurenhammer, F., *Voronoi Diagrams: a survey of a fundamental geometric data structure.* ACM Computing Surveys, 23:345-405, 1991.