The Point Set Order Type Data Base: A Collection of Applications and Results

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Abstract

Order types are a common tool to provide the combinatorial structure of point sets in the plane. For many problems in combinatorial and computational geometry only the order type of the underlying point set has to be considered. Recently a complete order type data base of n-point sets has been developed for $n \leq 10$, which gives a way to examine the combinatorial properties of all possible point sets for fixed size n. Based on this result we present applications and results for problems concerning intersection properties, convexity, crossing-free straight line graphs, and others, thus confirming or disproving several conjectures on these topics. Besides providing concrete results the aim of this work is to stimulate further research by revealing structural relations of extreme examples for 17 geometrical and combinatorial problems.

Keywords: Point configuration, order type, data base, geometric graph, crossing number, convex k-gon, convex cover, convex partition, convex decomposition, triangulation, pseudo-triangulation, simple polygonalization, spanning tree, matching, k-set.

1 Introduction

A finite point set in the Euclidean plane is the underlying structure for many problems in computational and combinatorial geometry. For many questions only their combinatorial properties have to be considered. More precisely, the crossing properties of the straight line edges spanned by the point set completely determine the situation. These intersection properties can be provided by the *order type* of a point set $\{p_1, ..., p_n\}$ which is defined as a mapping that assigns to each ordered triple i, j, k in $\{1, ..., n\}$ the orientation of the triple (p_i, p_j, p_k). Point sets of the same order type are called combinatorially equivalent. It is well known that two line segments spanned by any set intersect if and only if the two corresponding segments for an equivalent set do so. Thus for many interesting problems it is sufficient to deal only with inequivalent sets.

Recently a complete data base of inequivalent point sets in general position of size $n \leq 10$ has been established [3]. In the present work, applications of this data base to various problems are described. Questions concerning intersection properties, convexity, triangulations, polygonalization, and others are addressed. Tables of tight lower and upper bounds are given, providing several new results and answers to some open problems and conjectures. For most of the problems different approaches have been implemented yielding the same results, so high reliability of the data is guaranteed.

2 Applications and Results

For all instances and observations described in the following sections let S be a set of n points in the plane in general position, i.e. no three points of S lie on a common line. Let $E(S)$ be the set of straight line segments with endpoints in S, i.e. the edge set of the complete geometric graph K_n of S. Two segments of $E(S)$ are crossing if they intersect in a point other than one of their endpoints.

2.1 Intersection Properties

2.1.1 Rectilinear Crossing Number

Let $\overline{cr}(S)$ be the number of pairs of edges in $E(S)$ crossing each other. The minimum of $\overline{cr}(S)$ over all sets S of size n is the *rectilinear crossing number* $\overline{cr}(K_n)$, i.e. the number of crossings every complete straight line graph K_n of n points in the plane must have, see Figure 1.

Instance 1 *Determine the rectilinear crossing number $\overline{cr}(K_n)$.*

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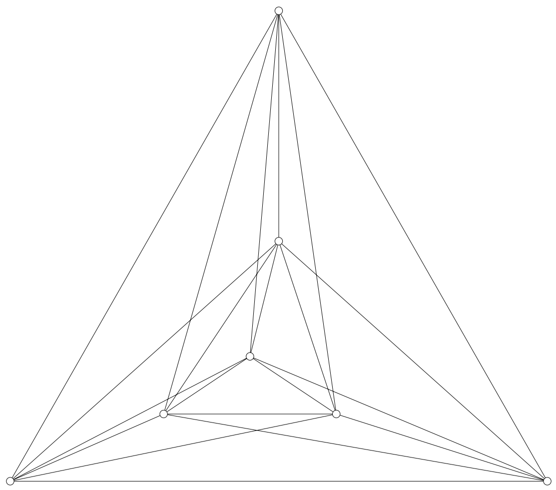

Figure 1: A drawing of K_7 with rectilinear crossing number $\overline{cr}(K_7) = 9$.

Several papers have dealt with crossing number problems before; see [8] for a historic overview.

Exact values of $\overline{cr}(K_n)$ have been known for $n \leq 9$ [15]. Erdös and Guy conjectured that $\overline{cr}(K_{10}) = 63$, but in 1971 it has been shown [28] that $\overline{cr}(K_{10})$ is either 61 or 62. Recently $\overline{cr}(K_{10}) = 62$ has been proven combinatorially [8]. Our complete enumeration independently revealed the same result:

Result 1 *Table 1 gives the exact rectilinear crossing numbers $\overline{cr}(K_n)$ for $n = 3, ..., 10$, in particular $\overline{cr}(K_{10}) = 62$.*

n	3	4	5	6	7	8	9	10
$\overline{cr}(K_n)$	0	0	1	3	9	19	36	62

Table 1: Rectilinear crossing number $\overline{cr}(K_n)$.

Several unsolved questions on rectilinear crossing numbers have been posed in [9]: What is the number of inequivalent sets leading to the same optimal crossing number $\overline{cr}(K_n)$? Especially only one set achieving $\overline{cr}(K_{10}) = 62$ was previously known [9]. Table 2 gives the exact numbers for $n \leq 10$. Note that there are two different drawings of K_{10} with crossing number $\overline{cr}(K_{10}) = 62$.

n	3	4	5	6	7	8	9	10
d_n	1	1	1	1	3	2	10	2

Table 2: Number d_n of inequivalent drawings of K_n with rectilinear crossing number $\overline{cr}(K_n)$.

A conjecture originally stated by Hayward [20] and restated in [9] provides a relation between the rectilinear crossing number and crossing-free Hamiltonian cycles (cfhc, cf. Section 2.4.1): Does some rectilinear drawing of K_n with the minimum number of edge crossings necessarily produce the optimal number of cfhc? Our examinations show that this conjecture is wrong in the general case. However, it still holds for $n \leq 8$, but for $n = 9$, none of the 10 point sets giving $\overline{cr}(K_9) = 36$ is the unique set maximizing the number of cfhc. The situation for $n = 10$ is similar.

Note that the *maximum* number $\overline{cr}(S)$ is attained by n points in convex position, giving $\overline{cr}(S) = \binom{n}{4}$. Moreover each crossing of K_n bijectively corresponds to a convex quadrilateral spanned by the underlying point set, see Section 2.2.2.

2.1.2 Crossing Families

A *crossing family* of size k of S is a set of k line segments from $E(S)$ which pairwise cross. A maximal crossing family of S is a crossing family of maximal size.

Instance 2 *Compute tight bounds for the maximal crossing family of an n-point set.*

From the fact that the complete graph with 5 vertices is not planar it follows that any set of $n \geq 5$ points has a crossing family of size at least 2. The following result on crossing families of size 3 improves over previously known bounds on n.

Result 2 *Every set S of $n \geq 10$ points admits a crossing family of size 3. The bound on n is tight, i.e. there exist 12 sets of 9 points which do have a maximal crossing family of size 2.*

Aronov et al. [6] proved the existence of crossing families of size $\sqrt{n/12}$ for every set of n points. This evaluates to $n \geq 108$ for size 3. A smaller bound, $n \geq 37$, derives from a result by Tóth and Valtr [30]: among any $\binom{2c-5}{c-2} + 2$ points there are at least c points in convex position. Thus there must be a crossing family of size $\lfloor \frac{c}{2} \rfloor$.

Result 2 plays a crucial role in deriving a new lower bound on the number of triangulations every planar point set must have, see Section 2.3.1 and [4].

2.2 Convexity

2.2.1 Extreme points

For several problems their examination might be restricted to sets with a fixed number of extreme points. For example, the triangulation isomorphism conjecture (cf. Section 2.3.2) is known to be true for a small number of interior points, thus only sets with a sufficiently

	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$	$\sum$
$n=3$	1	-	-	-	-	-	-	-	1
$n=4$	1	1	-	-	-	-	-	-	2
$n=5$	1	1	1	-	-	-	-	-	3
$n=6$	6	6	3	1	-	-	-	-	16
$n=7$	49	59	22	4	1	-	-	-	135
$n=8$	1 178	1 468	570	90	8	1	-	-	3 315
$n=9$	55 235	70 475	28 232	4 552	311	11	1	-	158 817
$n=10$	4 876 476	6 319 019	2 628 738	450 176	33 969	1 146	22	1	14 309 547

Table 3: Number of sets with n points, thereof h extreme points.

large number of interior points have to be considered. Other examples are sequential triangulations, see Section 2.3.3.

Instance 3 *Compute the number of sets with n points, h of which are extreme.*

Result 3 *Table 3 lists the numbers of point sets according to the size h of the convex hulls.*

2.2.2 Convex k-gons

The following two questions have been asked by Erdös and Guy [15]: What is the least number of convex quadrilaterals determined by n points in the plane? More generally, one can ask for the least number of convex k-gons determined by n points in the plane, for $k \geq 4$.

Instance 4 *Determine the least number of convex k-gons every set of n points must have.*

For 4-gons (convex quadrilaterals) this corresponds to the number of straight edge crossings, see Section 2.1.1. It was shown in [22] that every set of $n \geq 9$ points contains a convex 5-gon. Moreover it is necessary to consider $n \geq 2^{k-2}+1$ points in order to guarantee the existence of convex k-gons [14], and it is conjectured that this condition is also sufficient.

Result 4 *Table 4 gives the minimum number of convex k-gons every n-point set S determines, for $n \leq 10$ and $k = 3, 4, 5$. For $k \geq 6$ there are always sets without convex k-gons.*

n	3	4	5	6	7	8	9	10
3-gon	1	4	10	20	35	56	84	120
4-gon	-	0	1	3	9	19	36	62
5-gon	-	-	0	0	0	0	1	2

Table 4: Minimum numbers of convex k-gons.

Note that the maximum numbers of convex k-gons are obviously obtained by convex sets.

2.2.3 Empty Convex k-gon

How many *empty* convex k-gons does a set S contain? Again convex sets yield the maximum numbers, so we are only interested in the minimum numbers.

Instance 5 *Determine the least number of empty convex k-gons every set of n points must have.*

Result 5 *Table 5 gives the minimum number of empty convex k-gons, $k = 3, 4, 5$, every set S of size n contains. For $n \geq 5$ there always exists an empty convex 4-gon, and similarly for $n \geq 10$ there always exists an empty convex 5-gon. Both bounds are tight in n and k.*

n	3	4	5	6	7	8	9	10
3-gon	1	3	7	13	21	31	43	58
4-gon	-	0	1	3	6	10	15	23
5-gon	-	-	0	0	0	0	0	1

Table 5: Minimum number of empty convex k-gons.

There exist constructions of arbitrarily large point sets containing no empty convex 7-gon [21]. The question whether convex 6-gons can always be found in point sets of sufficiently large size n is still unresolved, but there is an example of 26 points that does not yield any convex 6-gon [26]. The existence of an empty convex 5-gon in every set of $n \geq 10$ points has already been proved by Harborth [19].

2.2.4 Convex Covering

For a point set S the *convex cover number* $\kappa_1(S)$ is the minimum number of convex polygons spanned by S and covering all points of S. The convex cover number for all sets of fixed size n is $\kappa_1(n) := max_S \kappa_1(S)$.

Instance 6 *Compute the convex cover numbers $\kappa_1(n)$.*

Result 6 *Table 6 gives the exact convex cover numbers $\kappa_1(n)$ for $n = 3, ..., 10$.*

Values of $\kappa_1(n)$ have been known for $n \leq 9$ [5]. See also the next section for more details.

n	3	4	5	6	7	8	9	10
$\kappa_1(n)$	1	2	2	2	2	2	3	3

Table 6: Convex cover number $\kappa_1(n)$.

2.2.5 Convex Partitioning

For a point set S the *convex partition number* $\kappa_2(S)$ is the minimum number of *disjoint* convex polygons spanned by S and covering all vertices of S. The convex partition number for all sets of fixed size n is $\kappa_2(n) := max_S \kappa_2(S)$.

Instance 7 *Compute the convex partition numbers* $\kappa_2(n)$.

Result 7 *Table 7 gives the exact convex partition numbers* $\kappa_2(n)$ *for* $n = 3, ..., 10$.

n	3	4	5	6	7	8	9	10
$\kappa_2(n)$	1	2	2	2	2	3	3	3

Table 7: Convex partition number $\kappa_2(n)$.

Obviously $\kappa_1(S) \leq \kappa_2(S)$, and it has been shown that $\kappa_2(S) \leq \rho(S) + 1$ [11], see Section 2.4.2 for a definition of $\rho(S)$. Values of $\kappa_2(n)$ have been known for $n \leq 9$.

2.2.6 Convex Decomposition

Convex decomposition is a well studied problem for simple polygons [23]. For a point set S a convex decomposition is a partition of the convex hull of S into convex polygons spanned by S. For instance every triangulation of S is a convex decomposition. The minimum number of polygons needed for a convex decomposition of S is the *convex decomposition number* $\kappa_3(S)$. The convex decomposition number for all sets of fixed size n is $\kappa_3(n) := max_S \kappa_3(S)$.

Instance 8 *Determine the convex decomposition numbers* $\kappa_3(n)$.

Result 8 *Table 8 gives the exact convex decomposition numbers* $\kappa_3(n)$ *for* $n = 3, ..., 10$.

n	3	4	5	6	7	8	9	10
$\kappa_3(n)$	1	3	4	6	7	8	10	11

Table 8: Convex decomposition number $\kappa_3(n)$.

It has been known that $n + 1$ is a lower bound for $\kappa_3(n)$, $n \geq 9$. Urrutia conjectured this bound to be tight, and provided the best known upper bound $\frac{3n}{2}$ [31]. Figure 2 gives a new lower bound example requiring at least $n + 2$ polygons, $n \geq 13$, for any convex decomposition. This disproves the conjecture.

Conjecture 1 *The exact convex decomposition number* $\kappa_3(n)$ *is* $n + 2$ *for* $n \geq 13$.

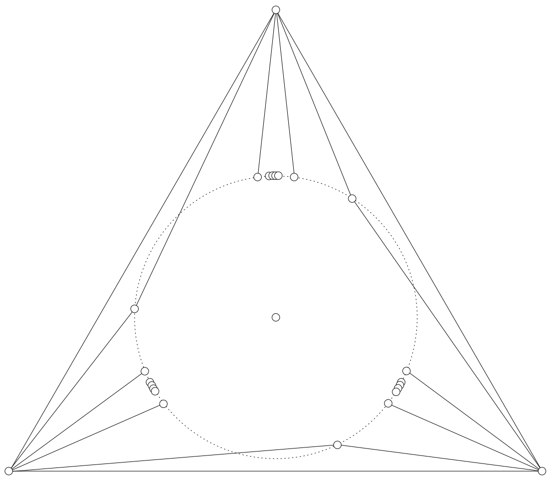

Figure 2: A class of n-point sets requiring at least $n+2$ polygons in any convex decomposition, $n \geq 13$. All inner points are cocircular. Overlapping points indicate point positions where an arbitrary number of points (including zero) might be placed.

2.3 Triangulations

2.3.1 Number of Triangulations

Computing the *number of triangulations* is a highly non-trivial task, and no polynomial-time methods have been developed so far. Even for point sets of small size no exact numbers have been known. Since combinatorially equivalent point sets admit the same number of triangulations these numbers have been counted for every point set in the data base.

Instance 9 *Compute exact upper and lower bounds on the number of triangulations of an* n*-point set.*

Applying brute-force methods turned out to be too time consuming for complete enumeration, thus rather sophisticated methods had to be used. To guarantee correctness of the results we used two different implementations of the divide and conquer like path-approach developed in [1]. Moreover to double-check the numbers another method, based on reverse search [7], was applied to the same sets, so the results have high reliability. Main results are tight lower and upper bounds:

	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$
$n=3$	**1/1**	-	-	-	-	-	-	-
$n=4$	**1**/1	2/**2**	-	-	-	-	-	-
$n=5$	**2**/2	3/3	5/**5**	-	-	-	-	-
$n=6$	**4**/8	6/8	9/11	14/**14**	-	-	-	-
$n=7$	**11**/30	13/32	19/34	28/36	42/**42**	-	-	-
$n=8$	32/**150**	**30**/135	43/144	62/127	90/127	132/132	-	-
$n=9$	96/**780**	**89**/690	102/645	145/584	207/488	297/435	429/429	-
$n=10$	305/**4550**	272/3809	**250**/3270	352/2788	497/2270	704/1806	1001/1549	1430/1430

Table 9: Exact minimum/maximum numbers of triangulations of n-point sets with h extreme points. Overall minimum/maximum numbers for fixed n are in bold type.

Result 9 *From $n = 3$ to 10, Table 9 gives tight lower and upper bounds on the number of triangulations of a set S of n points whereof h are extreme.*

Point sets in convex position give maximum numbers for $n \leq 7$, whereas for $n = 8$, 9, and 10 point sets with 3 extreme points maximize the number of triangulations. Minimum numbers are obtained by double circle sets (cf. [4]), which can be shown to have $(\sqrt{12})^{n-\Theta(\log(n))}$ triangulations [27]. Point sets with maximum number of triangulations are not unique in general, whereas the double circle sets are the unique minimizing sets. More details on point sets with extreme numbers of triangulations are available on the web [32].

Conjecture 2 *The maximum number of triangulations is obtained by point sets with 3 extreme points for $n \geq 8$.*

Conjecture 3 *The unique point sets minimizing the number of triangulations are double circle point sets.*

Table 9 together with Result 2 plays a crucial role in deriving an asymptotic lower bound of $\Omega(2+\epsilon)^n$, for some constant $\epsilon > 0$, on the number of triangulations every planar point set must have [4].

2.3.2 Isomorphism Conjecture for Planar Triangulations

Let S_1 and S_2 be two point sets, both of same cardinality and with equal-sized convex hulls $CH(S_1)$ and $CH(S_2)$. Two triangulations T_1 of S_1 and T_2 of S_2 are called compatible if the face lattices formed by their triangles, edges, and points are isomorphic, see Figure 3. The *isomorphism conjecture* for planar triangulations states that there exist compatible triangulations for any such pair S_1, S_2, see [2] for details.

Conjecture 4 *Let S_1 and S_2 be two sets of points without fixed correspondence. Then compatible triangulations for S_1 and S_2 exist if S_1 and S_2 fulfill the following properties:*

- $|S_1| = |S_2|$,
- $|CH(S_1)| = |CH(S_2)|$, *and*
- S_1 *and* S_2 *are in general position.*

Conjecture 5 *Conjecture 4 still holds if the correspondence between the points on the convex hulls of S_1 and S_2 is given by a cyclic labeling.*

Several classes of point sets are known to be universal, meaning the existence of compatible triangulations can be shown for any arbitrary second point set. For example point sets with at most 3 interior points are universal. Of course, the isomorphism conjecture implies that all point sets are universal.

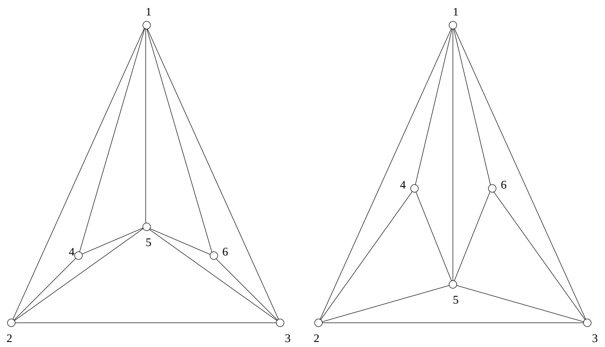

Figure 3: Unique compatible triangulations for two sets of 6 points.

Instance 10 *Search for isomorphic triangulations for all pairs of sets S_1, S_2.*

Result 10 *For $n \leq 8$ there exist isomorphic triangulations for all pairs S_1, S_2. This remains true even if the cyclic correspondence between the extreme points of S_1 and S_2 is prescribed.*

Currently several heuristics are investigated to make use of the data base for $n = 9$ (and in the long term for $n = 10$). Note that unlike for other applications all pairs of sets have to be considered (at least implicitly), i.e. the number of instances to be considered grows roughly quadratically with the number of sets.

2.3.3 Sequential Triangulations

A *sequential triangulation* of a point set S can be described by a sequence of point indices $i_1, i_2, ..., i_{t+2}$ of the underlying point set, where t is the number of triangles of any triangulation of the set. The triangles of the sequential triangulation are then defined by the point triples $p_{i_k}, p_{i_{k+1}}, p_{i_{k+2}}$ for $k = 1, ..., t$, see Figure 4 for an example.

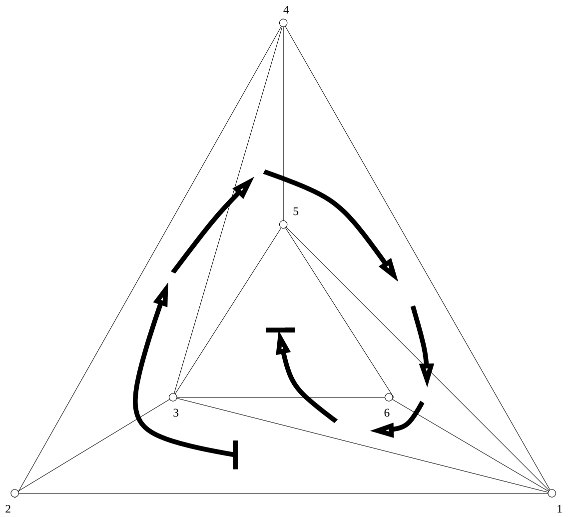

Figure 4: A sequential triangulation for a set of 6 points defined by the sequence $1, 2, 3, 4, 5, 1, 6, 3, 5$.

Although it is known that not all point sets admit sequential triangulations it is conjectured that for any n points whose convex hull is a triangle, there always exist sequential triangulations [25].

Conjecture 6 *Point sets with 3 extreme points always admit sequential triangulations.*

In other words: For any n points it is always possible to find a sequential triangulation of the entire plane (not just the convex hull, i.e. one additional point lies at infinity).

Instance 11 *Examine all sets S for the existence of sequential triangulations.*

Result 11 *Using the data base we obtained:*

- *For any point set with $n \leq 8$ points there exists a sequential triangulation.*
- *For $n = 9$ there exists exactly one set (with 6 extreme points) which has no sequential triangulation, and for $n = 10$ there are seven such sets, all with 7 extreme points.*
- *For any set S of size $n \leq 10$ whose convex hull is a triangle, there always exist sequential triangulations.*

It is interesting to note that two of the 10-point sets (and only these) can be reduced to the unique 9-point set by removing one point (in fact for one set there are three ways to do so). The other five sets are 'new' in this sense. The last part of our result strongly supports the conjecture mentioned above.

To examine all sets we used code implemented by Martin Held.

2.3.4 Minimum Pseudo-Triangulations

Pseudo-triangulations are structures with many combinatorial properties, and several applications [29]. A pseudo-triangle is a simple polygon that has exactly three convex vertices. A pseudo-triangulation of an n-point set S is a partition of the convex hull of S into pseudo-triangles whose vertex sets unite to S. Of particular interest are *minimum pseudo-triangulations* which minimize the number of pseudo-triangles. It is well known that minimum pseudo-triangulations consist of exactly $n - 2$ pseudo-triangles, see Figure 5 for an example. We are interested in bounds on the number of different minimum pseudo-triangulations which any n-point set admits.

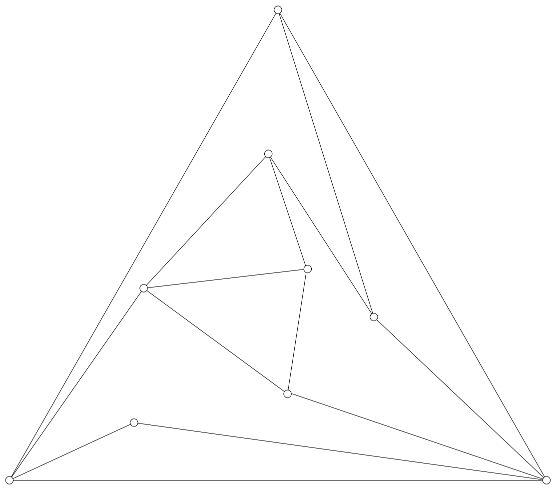

Figure 5: A minimum pseudo-triangulation for a set of 9 points consisting of 7 pseudo-triangles.

Instance 12 *Determine for every set S the number of minimum pseudo-triangulations.*

Result 12 *Table 10 gives exact lower and upper bounds on the number of minimum pseudo-triangulations of an n-point set for $n = 3, ..., 9$.*

n	min	max
3	1	1
4	2	3
5	5	13
6	14	76
7	42	485
8	132	3 555
9	429	27 874

Table 10: Minimum and maximum numbers of minimum pseudo-triangulations.

The implementations and computations on minimum pseudo-triangulations have been carried out by two independent groups ([10], [24]), providing the data for Table 10. For $n = 10$ the calculations are still ongoing.

We want to restate the following, common conjecture giving a connection between the number of pseudo-triangulations and the number of triangulations:

Conjecture 7 *For any given point set the number of pseudo-triangulations is at least the number of triangulations.*

Equality in Conjecture 7 is given for sets in convex position since there any pseudo-triangulation coincides with a triangulation. Combining the results on the number of triangulations, see Section 2.3.1, with the number of minimum pseudo-triangulations the conjecture has been verified for point sets with at most 9 points.

In contrary to triangulations the results show that the number of minimum pseudo-triangulations is minimized for sets in convex position for $n \leq 9$, where, as mentioned above, it coincides with the number of triangulations. We thus conjecture that this holds for arbitrary n.

Conjecture 8 *Point sets in convex position minimize the number of minimum pseudo-triangulations.*

2.4 Polygonalization

2.4.1 Crossing-Free Hamiltonian Cycles

A *crossing-free Hamiltonian cycle* (cfhc) of S is a closed chain of pairwise not-crossing segments of $E(S)$, including every point of S exactly once. One might ask how many cfhc do exist for a given set S. Obviously sets in convex position admit only one cfhc, so we are interested in the maximum numbers $\mathcal{HC}(n)$ of cfhc over all n-point sets. Hayward [20] cites prior work on $\mathcal{HC}(n)$ and improves the lower bound to $\mathcal{HC}(n) = \Omega(3.2684^n)$.

Instance 13 *Compute the maximum numbers $\mathcal{HC}(n)$ of cfhc over all sets S of size n.*

Result 13 *Table 11 gives exact upper bounds on the number of cfhc of sets S with $n = 3, ..., 10$ points.*

n	3	4	5	6	7	8	9	10
$\mathcal{HC}(n)$	1	3	8	29	92	339	1282	4994

Table 11: Maximum number $\mathcal{HC}(n)$ of cfhc.

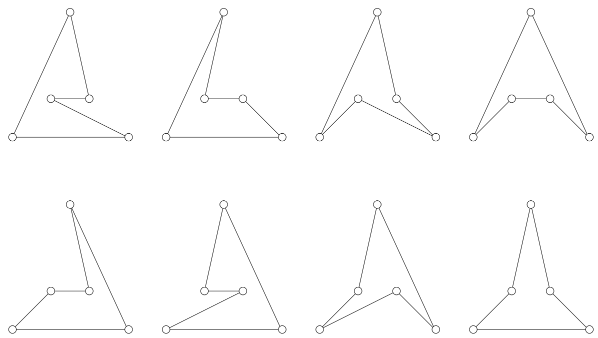

Figure 6: 5 points with maximum number $\mathcal{HC}(5) = 8$.

Exact numbers for $\mathcal{HC}(n)$ have been known for $n \leq 6$ [20]. For $n = 7, 8$ known lower bounds were found to be tight and for $n = 9, 10$ our calculations gave new and tight bounds. Moreover we observed that all sets with maximum number $\mathcal{HC}(n)$ are unique for $n \leq 10$, and have convex hulls of size 3. We thus conjecture:

Conjecture 9 *Point sets with maximum number $\mathcal{HC}(n)$ of cfhc are unique and have 3 extreme points.*

2.4.2 Minimizing the Number of Reflex Angles in Simple Polygonalizations

A cfhc is also called a *polygonalization*. A natural question is how 'convex' the best polygonalization of a point set S can be, i.e. what is the minimal number $\rho(S)$ of reflex angles a polygonalization of S can have? More generally, what is the minimum number $\rho(n)$ such that every point set S with n points has a polygonalization with at most $\rho(n)$ reflex angles? See [5] for recent results and more details.

Instance 14 *Determine $\rho(n)$.*

Values for $\rho(n)$ are found by counting the number of reflex points for all cycles generated in Section 2.4.1, computing $\rho(S)$ and taking $\rho(n) := max_{S,|S|=n}\rho(S)$.

Result 14 *For $n = 3, ..., 10$ Table 12 gives the exact values for $\rho(n)$.*

It can be shown that $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$, and values of $\rho(n)$ have been known for $n \leq 9$ [5].

n	3	4	5	6	7	8	9	10
$\rho(n)$	0	1	1	2	2	2	3	3

Table 12: Tight upper bound $\rho(n)$ of reflex angles in the 'best' polygonalization of any n points.

2.5 More Crossing-Free Graphs

2.5.1 Crossing-Free Spanning Trees

A (straight line) *crossing-free spanning tree* of S is a set of $n-1$ non-intersecting segments of $E(S)$, connecting all points in S. A natural question is how many such trees a point set does have. It is known that this number is minimized by points in convex position [17]. A special configuration of points leads to $\Omega(10.42^n)$ crossing-free spanning trees [13]. However, there are no tight upper bounds.

Instance 15 *Compute the maximum number of crossing-free spanning trees.*

Result 15 *For* $n = 3, \ldots, 10$ *Table 13 gives tight lower and upper bounds on the number of crossing-free spanning trees.*

n	min	max
3	3	3
4	12	16
5	55	105
6	273	891
7	1 428	7 815
8	7 752	77 953
9	43 263	819 822
10	246 675	9 179 632

Table 13: Tight lower and upper bounds on the number of straight line crossing-free spanning trees.

All point sets with minimum and maximum number of crossing-free spanning trees are unique for $n \leq 10$. In addition the convex hulls of maximizing sets are triangles, and therefore we conjecture:

Conjecture 10 *Point sets with maximum number of crossing-free spanning trees are unique and have* 3 *extreme points.*

2.5.2 Crossing-Free Matchings

A (straight line) *crossing-free matching* in a set S of n points is a set of $\lfloor \frac{n}{2} \rfloor$ pairwise not intersecting segments of $E(S)$, having no endpoints in common. Here the question is how many such matchings a point set exhibits. The minima are obtained by convex sets [17]. Minimum numbers are $C_{\frac{n}{2}}$ for n even, and $n \cdot C_{\frac{n-1}{2}}$ for n odd, where C_n denotes the n-th Catalan number.

Instance 16 *Determine the maximum number of crossing-free matchings.*

Result 16 *For* $n = 3, \ldots, 10$ *Table 14 gives tight lower and upper bounds on the number of crossing-free matchings.*

n	3	4	5	6	7	8	9	10
min	3	2	10	5	35	14	126	42
max	3	3	14	12	79	56	497	311

Table 14: Tight lower and upper bounds on the number of crossing-free matchings.

2.6 Miscellaneous

2.6.1 k-Sets

For a given set S, a *k-set* is a k-element subset of S that can be separated from S by a straight line. It is not hard to see that the minimum number of k-sets is $2k+1$ for $k < \frac{n}{2}$, and k for $k = \frac{n}{2}$. Finding bounds on the maximum number $f_k(n)$ of k-sets is an intriguing problem which has been studied extensively; see, e.g., Dey [12] and references therein.

Instance 17 *Compute the maximum number* $f_k(n)$ *of* k*-sets.*

Result 17 *For* $n = 3, \ldots, 10$ *Table 15 gives exact values for the maximum number* $f_k(n)$ *of* k*-sets.*

n	3	4	5	6	7	8	9	10
1-sets	3	4	5	6	7	8	9	10
2-sets	-	3	7	9	10	12	13	15
3-sets	-	-	-	6	12	14	16	18
4-sets	-	-	-	-	-	9	18	21
5-sets	-	-	-	-	-	-	-	13

Table 15: Exact values for the maximum number $f_k(n)$ of k-sets.

Recently Felsner [16] determined the maximum number of $\frac{n}{2}$-sets (halving lines) for 10 points, $f_5(10) = 13$, which confirms our result.

3 Concluding Remarks

We have successfully applied the order type data base to several problems in computational and combinatorial geometry. Our work already exhibits various new

index	$\lvert CH \rvert$	characteristics and remarks
1	10	**convex set;** max crossing number; min mpt; min cfhc; min cf sp.trees; min cf match.; max 1-sets
4 810	7	no sequential triangulation
56 717	7	no sequential triangulation
61 150	7	no sequential triangulation
136 834	7	no sequential triangulation
136 894	7	no sequential triangulation
137 811	7	no sequential triangulation
163 303	7	no sequential triangulation
2 653 259	4	max 5-sets
10 249 535	3	max 5-sets
12 492 616	3	max 5-sets
12 698 079	4	max 3-sets
12 700 934	3	max 3-sets
12 700 989	3	max 3-sets
12 723 664	3	max 3-sets
13 413 894	3	min crossing number; max triangulations; max cf spanning trees; max 5-sets
13 468 936	3	max cf Hamiltonian cycles
13 501 926	3	max 4-sets
13 789 280	3	max cf matchings
13 812 360	3	min crossing number; max triangulations; max 5-sets
14 088 800	5	**double circle;** min triangulations; max 2-sets

Table 16: Overview of sets with extremal properties for $n = 10$ points.

results, but there are still more interesting problems to be investigated.

To provide more insight into the structural relation between extreme examples of different problems Table 16 shows the indices of point sets minimizing or maximizing certain properties for $n = 10$. If there are no indices given for a specific problem this means that more than 10 sets achieve the related extrema. The indices are taken from the order type data base reflecting the sorted order according to the lexicographically minimized λ-matrices, see [3] for details. For example, it is not hard to see that the points of the first set are in convex position.

Not surprisingly the convex set is known to optimize a variety of properties. Apart from that the most interesting structure is given by set no. 13 413 894. As for set no. 13 812 360 it minimizes the crossing number $\overline{cr}(K_{10})$, maximizes the number of triangulations, maximizes the number of halving sets, and is in addition the unique set maximizing the number of crossing-free spanning trees. Similar relations exist for sets of size 8, and with the exception of maximizing the number of triangulations for smaller sets with even cardinality. This provides sufficient evidence to make us believe that the common structure of extremal sets between these loosely related problems can be generalized to larger instances, at least for even cardinality.

As mentioned in Section 2.3.1 the double circle minimizes the number of triangulations. Moreover, it provides the maximum number of 2-sets, which can be shown to hold in general.

Above mentioned examples are by no means exhaustive, actually we expect more interesting results to show up. To this end we plan to provide a tool for online structured data base queries [33].

We hope that both, shown and still to be encountered observations, stimulate further research on the presented topics. Therefore to enable the interested reader to perform her/his own investigations all sets and our results are made public on the web [33]. Due to space limitations, the grid point sets of size 10 are not accessible online. They are available on a CD upon request.

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