The Point Set Order Type Data Base: A Collection of Applications and Results (Extended Abstract)

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1 Introduction

A finite point set in the Euclidean plane is the underlying structure for many problems in computational and combinatorial geometry. For many questions only their combinatorial properties have to be considered. More precisely, the crossing properties of the straight line edges spanned by the point set completely determine the situation. These intersection properties can be provided by the *order type* of a point set $\{p_1, ..., p_n\}$ which is defined as a mapping that assigns to each ordered triple i, j, k in $\{1, ..., n\}$ the orientation of the triple (p_i, p_j, p_k). Point sets of the same order type are called combinatorially equivalent. It is well known that two line segments spanned by any set intersect if and only if the two corresponding segments for an equivalent set do so. Therefore for many interesting problems it is sufficient to deal only with inequivalent sets. Recently a complete data base of inequivalent point sets in general position of size $n \leq 10$ has been established [2]. In the present work, applications of this data base to various problems are described. Questions concerning intersection properties, convexity, triangulations, polygonalization, and others are addressed. Tables of tight lower and upper bounds are given, providing several new results and answers to some open problems and conjectures. For most of the problems different approaches have been implemented yielding the same results, so high reliability of the data is guaranteed.

2 Applications and Results

For all instances and observations described in the following sections let S be a set of n points in the plane in general position, i.e. no three points of S lie on a common line. Let $E(S)$ be the set of straight line segments with endpoints in S. Two segments of $E(S)$ cross if they properly intersect in their interior.
In the following sections we give a brief overview of several investigated problems. More details and further results are given in the full paper. For consistency with the full paper the section and problem numbers may contain holes.

2.1 Intersection Properties

2.1.1 Rectilinear Crossing Number

Let $\overline{cr}(S)$ be the number of pairs of edges in $E(S)$ crossing each other. The minimum of $\overline{cr}(S)$ over all sets S of size n is the *rectilinear crossing number* $\overline{cr}(K_n)$.

Instance 1 *Determine the rectilinear crossing number* $\overline{cr}(K_n)$.

Exact values of $\overline{cr}(K_n)$ have been known for $n \leq 9$, and it was conjectured that $\overline{cr}(K_{10}) = 63$ [7]. Our complete enumeration disproved this conjecture:

Result 1 *Table 1 gives rectilinear crossing numbers* $\overline{cr}(K_n)$ *for* $n = 3, ..., 10$, *in particular* $\overline{cr}(K_{10}) = 62$.

n	3	4	5	6	7	8	9	10
$\overline{cr}(K_n)$	0	0	1	3	9	19	36	62

Table 1: Rectilinear crossing number $\overline{cr}(K_n)$.

2.1.2 Crossing Families

A *crossing family* of size k of S is a set of k line segments from $E(S)$ which pairwise cross. A maximal crossing family is a crossing family of maximal size.

Instance 2 *Compute the maximal crossing families.*

From the fact that the complete graph with 5 vertices is not planar it follows that any set of $n \geq 5$ points has a crossing family of size at least 2. The following result on crossing families of size 3 improves over previously known bounds on n, e.g. $n \geq 37$ derives from a result in [13].

*Research supported by APART [Austrian Programme for Advanced Research and Technology] of the Austrian Academy of Sciences

†Research supported by the FWF [Austrian Fonds zur Förderung der Wissenschaftlichen Forschung]

Result 2 *Every set S of $n \geq 10$ points admits a crossing family of size 3. The bound on n is tight, i.e. there exist sets of 9 points which do have a maximal crossing family of size 2.*

2.2 Convexity

2.2.2 Convex k-gons

The following questions have been asked by Erdös and Guy [7]: What is the least number of convex k-gons determined by n points in the plane?

Instance 4 *Determine the least number of convex k-gons every set of n points must have.*

For 4-gons (convex quadrilaterals) this corresponds to the number of straight edge crossings, see 2.1.1. It was shown in [12] that every set of $n \geq 9$ points contains a convex 5-gon.

Result 4 *Table 2 gives the minimum number of convex k-gons every set S of n points determines.*

n	3	4	5	6	7	8	9	10
3-gon	1	4	10	20	35	56	84	120
4-gon	-	0	1	3	9	19	36	62
5-gon	-	-	0	0	0	0	1	2

Table 2: Minimum numbers of convex k-gons.

2.2.3 Empty Convex k-gon

How many empty convex k-gons does a set S contain?

Instance 5 *Determine the least number of empty convex k-gons every set of n points must have.*

Result 5 *Table 3 gives the minimum number of empty convex k-gons, $k = 3, 4, 5$, every set S of size n contains. For $n \geq 5$ there always exists an empty convex 4-gon, and similarly for $n \geq 10$ an empty convex 5-gon.*

n	3	4	5	6	7	8	9	10
3-gon	1	3	7	13	21	31	43	58
4-gon	-	0	1	3	6	10	15	23
5-gon	-	-	0	0	0	0	0	1

Table 3: Minimum numbers of empty convex k-gons.

There exist constructions of arbitrarily large point sets containing no empty convex 7-gon [11]. The question whether convex 6-gons can always be found in point sets of sufficiently large size n is still unresolved. The existence of an empty convex 5-gon in every set of $n \geq 10$ points has already been proven in [9].

2.2.4 Convex Covering

For a point set S the *convex cover number* $\kappa_1(S)$ is the minimum number of convex polygons spanned by S and covering all points of S. The convex cover number for all sets of fixed size n is $\kappa_1(n) := max_S \kappa_1(S)$.

Instance 6 *Compute the convex cover numbers $\kappa_1(n)$.*

Result 6 *Table 4 gives the exact convex cover numbers $\kappa_1(n)$ for $n = 3, ..., 10$.*

Values of $\kappa_1(n)$ have been known for $n \leq 9$ [4].

n	3	4	5	6	7	8	9	10
$\kappa_1(n)$	1	2	2	2	2	2	3	3
$\kappa_2(n)$	1	2	2	2	2	3	3	3

Table 4: Convex cover number $\kappa_1(n)$ and convex partition number $\kappa_2(n)$.

2.2.5 Convex Partitioning

For a point set S the *convex partition number* $\kappa_2(S)$ is the minimum number of disjoint convex polygons spanned by S and covering all vertices of S. The convex partition number for all point sets of fixed size n is $\kappa_2(n) := max_S \kappa_2(S)$.

Instance 7 *Compute convex partition numbers $\kappa_2(n)$.*

Result 7 *Table 4 gives the exact convex partition numbers $\kappa_2(n)$ for $n = 3, ..., 10$.*

Values of $\kappa_2(n)$ have been known for $n \leq 9$ [4].

2.3 Triangulations

2.3.1 Number of Triangulations

Computing the *number of triangulations* is a highly non-trivial task, and no polynomial-time methods have been developed so far. Even for point sets of small size no exact bounds have been known.

Instance 9 *Compute tight upper and lower bounds on the number of triangulations of an n-point set.*

Result 9 *From $n = 3$ to 10, Table 5 gives tight lower and upper bounds on the number of triangulations of a set S of n points.*

n	3	4	5	6	7	8	9	10
min	1	1	2	4	11	30	89	250
max	1	2	5	14	42	150	780	4550

Table 5: Exact extremum numbers of triangulations.

Table 5 together with Result 2 plays a crucial role in deriving an asymptotic lower bound on the number of triangulations every planar point set must have [3].

2.3.2 Isomorphism Conjecture for Planar Triangulations

Let S_1 and S_2 be two point sets, both of same cardinality and with equal-sized convex hulls. The *isomorphism conjecture* for planar triangulations states that there exist isomorphic triangulations for any such pair S_1, S_2, see [1] for more details.

Instance 10 *Search for isomorphic triangulations for all pairs of sets S_1, S_2.*

Result 10 *For $n \leq 8$ there exist isomorphic triangulations for all pairs S_1, S_2. This remains true even if the cyclic correspondence between the extreme points of S_1 and S_2 is prescribed.*

Currently several heuristics are tested to make use of the data base for $n = 9$. Note that unlike for other applications all pairs of sets have to be considered.

2.3.3 Sequential Triangulations

A *sequential triangulation* of a point set S can be described by a sequence of point indices $i_1, i_2, ..., i_{t+2}$ of the underlying point set, where t is the number of triangles of any triangulation of the set [5]. The triangles of the sequential triangulation are then defined by point triples $p_{i_k}, p_{i_{k+1}}, p_{i_{k+2}}$ for $k = 1, ..., t$.

Instance 11 *Examine all sets S for the existence of sequential triangulations.*

Result 11 *For any point set with $n \leq 8$ points there exists a sequential triangulation. For $n = 9$ there exists exactly one set which has no sequential triangulation (with 6 extreme points), and for $n = 10$ there are seven such sets, all with 7 extreme points.*

To examine all sets we used code implemented by Martin Held.

2.4 Polygonalization

2.4.1 Crossing-Free Hamiltonian Cycles

A *crossing-free Hamiltonian cycle* (cfhc) of S is a closed chain of pairwise not crossing segments of $E(S)$, including every point of S exactly once. We are interested in the maximum numbers $\mathcal{HC}(n)$ of cfhc over all n-point sets.

Instance 13 *Compute the maximum numbers $\mathcal{HC}(n)$ of cfhc over all sets S of size n.*

Result 13 *Table 6 gives tight upper bounds on the number of cfhc of sets S with $n = 3, ..., 10$ points.*

Exact numbers for $\mathcal{HC}(n)$ have been known for $n \leq 6$ [10]. For $n = 7, 8$ known lower bounds were found to be tight and for $n = 9, 10$ our calculations gave new and tight bounds.

n	3	4	5	6	7	8	9	10
$\mathcal{HC}(n)$	1	3	8	29	92	339	1282	4994

Table 6: Maximum number $\mathcal{HC}(n)$ of cfhc.

2.4.2 Minimizing the Number of Reflex Angles in Simple Polygonalizations

A cfhc is also called *polygonalization*. A natural question is how 'convex' the best polygonalization of a point set S can be, i.e. what is the minimal number $\rho(S)$ of reflex angles a polygonalization of S can have? More generally, what is the minimum number $\rho(n)$ such that every point set S with n points has a polygonalization with at most $\rho(n)$ reflex angles? See [4] for recent results.

Instance 14 *Determine $\rho(n)$.*

Values for $\rho(n)$ are found by counting the number of reflex points for all cycles found in Section 2.4.1, computing $\rho(S)$ and taking $\rho(n) := max_{S,|S|=n}\rho(S)$.

Result 14 *For $n = 3, ..., 10$ Table 7 gives the exact values for $\rho(n)$.*

n	3	4	5	6	7	8	9	10
$\rho(n)$	0	1	1	2	2	2	3	3

Table 7: Tight upper bound $\rho(n)$ of reflex angles in the 'best' polygonalization of any n points.

It is known that $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$, and values of $\rho(n)$ have been known for $n \leq 9$ [4].

2.5 More Crossing-Free Graphs

2.5.1 Crossing-Free Spanning Trees

A (straight line) *crossing-free spanning tree* of S is a set of $n - 1$ non-intersecting segments of $E(S)$, connecting all points in S. A natural question is how many such trees a point set does have. It is known that this number is minimized by points in convex position [8]. However, there are no tight upper bounds.

Instance 15 *Compute the maximum number of crossing-free spanning trees.*

Result 15 *For $n = 3, ..., 10$ Table 8 gives tight lower and upper bounds on the number of crossing-free spanning trees.*

2.5.2 Crossing-Free Matchings

A (straight line) *crossing-free matching* in a set S of n points is a set of $\lfloor \frac{n}{2} \rfloor$ pairwise not intersecting segments of $E(S)$, having no endpoints in common. Here the question is how many such matchings a point set exhibits. The minima are obtained by convex sets [8].

n	min	max
3	3	3
4	12	16
5	55	105
6	273	891
7	1 428	7 815
8	7 752	77 953
9	43 263	819 822
10	246 675	9 179 632

Table 8: Tight lower and upper bounds on the number of straight line crossing-free spanning trees.

Instance 16 *Determine the maximum number of crossing-free matchings.*

Result 16 *For $n = 3, ..., 10$ Table 9 gives tight bounds on the number of crossing-free matchings.*

n	3	4	5	6	7	8	9	10
min	3	2	10	5	35	14	126	42
max	3	3	14	12	79	56	497	311

Table 9: Tight lower and upper bounds on the number of crossing-free matchings.

2.6 Miscellaneous

2.6.1 k-Sets

For a given set S a *k-set* is a k-element subset of S that can be separated from S by a straight line. Finding bounds on the maximum number $f_k(n)$ of k-sets is an intriguing problem which has been studied extensively; see, e.g., Dey [6] and references therein.

Instance 17 *Compute the maximum number $f_k(n)$ of k-sets.*

Result 17 *For $n = 3, ..., 10$ Table 10 gives exact values for the maximum number $f_k(n)$ of k-sets.*

n	3	4	5	6	7	8	9	10
1-sets	3	4	5	6	7	8	9	10
2-sets	-	3	7	9	10	12	13	15
3-sets	-	-	-	6	12	14	16	18
4-sets	-	-	-	-	-	9	18	21
5-sets	-	-	-	-	-	-	-	13

Table 10: Exact values for the maximum number $f_k(n)$ of k-sets.

3 Concluding Remarks

We have successfully applied the order type data base to several problems in computational and combinatorial geometry. Our work provides various new results and observations. More problems are considered in the full paper like extreme points, convex decomposition, and minimum pseudo-triangulations.

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