Complete Combinatorial Generation of Small Point Configurations and Hyperplane Arrangements

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Abstract

A recent progress on the complete enumeration of oriented matroids enables us to generate all combinatorial types of small point configurations and hyperplane arrangements in general dimension, including degenerate ones. This extends a number of former works which concentrated on the nondegenerate case and are usually limited to dimension 2 or 3. Our initial study on the complete list for small cases has shown its potential in resolving geometric conjectures.

1 Introduction

The generation of combinatorial types of point configurations and hyperplane arrangements has long been an outstanding problem of combinatorial geometry. A point configuration is a set of n points in the real Euclidean space $\mathbb{R}^d$. Its combinatorial type, called *order type*, is determined by the relative positions of the points, more formally by the set of all partitions of the n points by hyperplanes, where the points may be arbitrarily relabeled. Similarly, a hyperplane arrangement is a set of n affine hyperplanes in $\mathbb{R}^d$, and its combinatorial type, which we call its *dissection type*, is determined by the relative positions of all cells. For the generation of these combinatorial types no direct method is known, and it appears to be necessary to use combinatorial abstractions — allowable sequences of permutations, λ-functions, chirotopes, combinatorial geometries, or oriented matroids; in our work we will use *oriented matroids* [BLVS$^+$99]. These abstractions are more general than their geometric counterparts, e.g., there exist oriented matroids which cannot be *realized* by any point configuration. Although it is NP-hard to decide whether a given oriented matroid is realizable or not [Mnëv88, Sho91] the realizability problem is decidable and there are practical methods which work satisfactory for small instances (e.g., see [RG92]).

The former work on generation of point configurations and related structures (e.g., see [GP83, BGdO00, AAK01]) concentrated on the special cases of nondegenerate configurations (i.e., the cases where e.g. no three points lie on a line) and low dimensions (i.e., $d = 2$ or $d = 3$). We generate the entire list of all cases for small n, including degenerate cases in arbitrary dimension d. This complete generation of small point configurations and hyperplane arrangements offers a powerful database for various investigations as we will show by some examples.

2 Combinatorial Types and Oriented Matroids

We explain in this section how order types and dissection types relate to oriented matroids which are here illustrated by sphere arrangements.

Consider a point configuration $X = \{x^1, \ldots, x^n\}$ in $\mathbb{R}^d$. Let a hyperplane $H = H_v$ be described by a vector $v \in \mathbb{R}^{d+1}$ and the function $H_v(x) = \sum_{i=1}^{d} v_i x_i + v_{d+1}$ such that the sign of $H_v(x)$ defines the partition of X by H. Then the order type of X is determined by the set $\mathcal{V}(X)$ of all sign vectors $V \in \{-, +, 0\}^n$ such that V_e is the sign of $H_v(x^e)$ for all $e = 1, \ldots, n$ for some $v \in \mathbb{R}^{d+1}$. It is hence natural to embed X in $\mathbb{R}^{d+1}$ by adding $x^e_{d+1} = 1$ to every x^e as then $H_v(x)$ becomes the scalar product of v and x in $\mathbb{R}^{d+1}$. Geometrically, we can consider the extended vectors from X as the normal vectors of a central arrangement of hyperplanes, and this intersected with the unit sphere S^d leads to a sphere arrangement as depicted in Figure 1. Every sphere in the arrangement has an orientation according to the corresponding normal vector, and by this every cell in the sphere arrangement has a one-to-one relation to a sign vector in $\mathcal{V}(X)$ as introduced above. In the theory of oriented matroids these cell complexes are well studied; in fact every realizable oriented matroid has a representation by a sphere arrangement, and every oriented matroid a representation by a topological sphere arrangement.

An oriented matroid which is defined by a point configuration X has the special property that the sign vector

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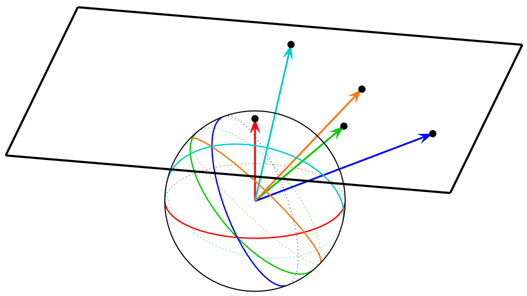

Figure 1: Point configuration and sphere arrangement

$(+\cdots+)$ is in $\mathcal{V}(X)$: $(+\cdots+)$ corresponds to the cell containing $v=(0,\ldots,0,1)$. We call the relabeling class of such an oriented matroid an *abstract order type*.

Consider now a hyperplane arrangement $\{h^1,\ldots,h^n\}$ in $\mathbb{R}^d$. Similarly as for point configurations, we embed this arrangement in $\mathbb{R}^{d+1}$ by fixing the new coordinate to be 1. Every h^e determines a hyperplane H^e in $\mathbb{R}^{d+1}$ which contains h^e and the origin $0\in\mathbb{R}^{d+1}$. All H^e intersected with the unit sphere S^d lead again to a sphere arrangement — where the orientations of the spheres are not given and may be chosen arbitrarily. This sphere arrangement corresponds to a projective hyperplane arrangement; for the given Euclidean hyperplane arrangement we have to add information how it was projected onto S^d, and we can do this by adding an extra sphere with normal vector $(0,\ldots,0,1)$ which is specially marked (see Figure 2, the extra sphere is

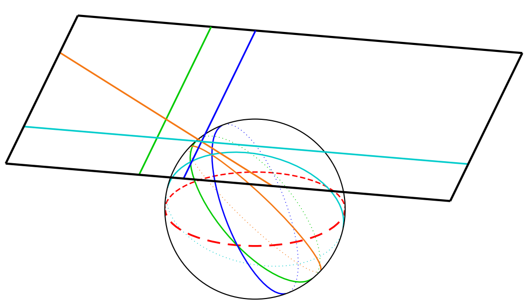

Figure 2: Hyperplane arrangement and sphere arrangement

dashed). Hence oriented matroids which are defined by hyperplane arrangements have the special property that one element is specially marked: We call this the *infinity* element. Note that the orientation of the sphere arrangement is arbitrary and therefore we say that two oriented matroids define the same *abstract dissection type* if they coincide for some choice of orientation and some relabeling which identifies the infinity elements.

3 Generation of Combinatorial Types

We will generate order types and dissection types from oriented matroids, using the relation which we discussed in Section 2. For this let be n and d given and consider the complete list $I(n,d)$ of all oriented matroids up to isomorphism, i.e., up to reorientation and relabeling; using the model of Section 2 we may think of $I(n,d)$ as a list containing all types of unlabeled and unoriented topological sphere arrangements with n spheres on S^d. We describe in [FF01] several methods how $I(n,d)$ can be generated. We will use $I(n,d)$ as the input for the following. Table 1 shows the numbers $|I(n,d)|$ of oriented matroids up to isomorphism, and for comparison we give in Table 2 the corresponding numbers for the nondegenerate cases only (these have been generated and discussed in former work).

$n=$	3	4	5	6	7	8	9
$d=2$	1	2	4	17	143	4890	461053
$d=3$		1	3	12	206	181472	
$d=4$			1	4	25	6029	
$d=5$				1	5	50	508321
$d=6$					1	6	91
$d=7$						1	7
$d=8$							1

Table 1: Oriented matroids up to isomorphism

$n=$	3	4	5	6	7	8	9
$d=2$	1	1	1	4	11	135	4382
$d=3$		1	1	1	11	2628	
$d=4$			1	1	1	135	
$d=5$				1	1	1	4382
$d=6$					1	1	1
$d=7$						1	1
$d=8$							1

Table 2: Oriented matroids up to isomorphism: nondegenerate cases only

Abstract order types have the special property that some cell in the oriented sphere arrangement corresponds to the sign vector $(+\cdots+)$. Consider any oriented sphere arrangement A in $I(n,d)$, in A some cell c of maximal dimension and its corresponding sign vector $V(c)$: A reorientation of A according to $V(c)$ will let c correspond to $(+\cdots+)$. Hence the list of all sign vectors corresponding to cells of maximal dimension in A, which we can compute efficiently, is sufficient to find all abstract order types isomorphic to A. Tables 3 shows the numbers of abstract order types obtained by computations (note that there are considerably fewer *non-degenerate* abstract order types, e.g., 1, 2, 3, 16, 135, 3315, 158830 for $d = 2$ and $n = 3, \dots, 9$).

$n =$	3	4	5	6	7	8	9
$d = 2$	1	3	11	93	2121	122508	15296266
$d = 3$		1	5	55	5083	10775236	
$d = 4$			1	8	204	505336	
$d = 5$				1	11	705	
$d = 6$					1	15	2293
$d = 7$						1	19
$d = 8$							1

Table 3: Abstract order types

The complete list of abstract dissection types for n hyperplanes in $\mathbb{R}^d$ is obtained from $I(n+1, d)$: For every sphere arrangement A in $I(n+1, d)$ there are $n+1$ choices for the infinity element which leads to all abstract dissection types isomorphic to A. The numbers of abstract dissection types obtained by computations can be found in Table 4.

$n+1 =$	3	4	5	6	7	8	9
$d = 2$	1	3	8	46	790	37829	4134939
$d = 3$		1	5	27	1063	1434219	
$d = 4$			1	7	71	44956	
$d = 5$				1	9	156	
$d = 6$					1	11	325
$d = 7$						1	13
$d = 8$							1

Table 4: Abstract dissection types

As remarked before, not all oriented matroids are realizable. In the nondegenerate case, the realizability problem is solved for $n \leq 8$ ($n \leq 10$ for $d = 2$), where the problem is attacked from two sides: (i) finding realizations (using randomly generated points, various insertion or perturbation techniques) and (ii) proving that no realization can exist (e.g., with final polynomials). The general case still needs work in both directions: Finding coordinates has the additional difficulty that some realizable instances do not have rational solutions; on the other hand some of the earlier methods to detect non-realizability have to be generalized to the degenerate case.

We present in Figures 3 and 4 realizations of abstract order types and abstract dissection types for small instances in $\mathbb{R}^2$, i.e., for configurations of 5 points and arrangements of 4 hyperplanes (here, all abstract combinatorial types can be realized). The trivial types of colinear points (i.e., all points on a line) and all hyperplanes parallel correspond to combinatorial types in $\mathbb{R}^1$ and are not counted in $\mathbb{R}^2$.

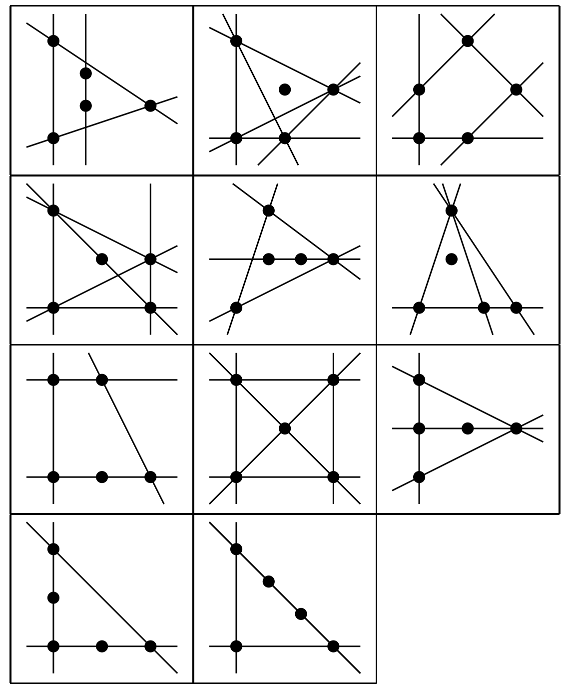

Figure 3: The 11 order types with 5 non-colinear points in $\mathbb{R}^2$; only the first 3 are non-degenerate

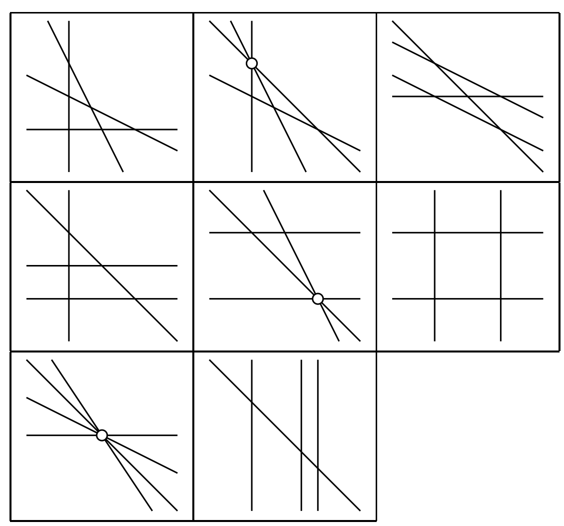

Figure 4: The 8 dissection types with 4 non-parallel hyperplanes in $\mathbb{R}^2$; only the first is non-degenerate

4 Applications

Before we discuss one example in more detail below, the following few remarks may hint on some possible impacts of our database of combinatorial types. There has been a strong interest in the number of faces (f-vectors) and specially the number of simplicial topes (mutations) of arrangements and oriented matroids, or in k-sets and extremal properties of point configurations; our database provides all these data for further investigations.

The list of abstract order types has been used to compute all (abstract) types of convex polytopes which coincide with the known numbers of combinatorial types of polytopes (e.g., see [KK95]), by this providing an independent proof of these results.

Consider now the following conjecture of Da Silva and Fukuda (Conjecture 4.2 in [DSF98]), which is a strong version of the Sylvester-Gallai Theorem: *Let X be a point configuration in $\mathbb{R}^2$, not all points on a line. Let H be a line which does not contain a point from X but separates X into two parts H^- and H^+ such that $|H^-|$ and $|H^+|$ differ by at most 1. Then there exists a line $\tilde{H}$ which contains exactly two points of X, one from H^- and one from H^+.*

Some weaker versions of this conjecture have been proved by Pach and Pinchasi [PP00]. We have tested the conjecture itself against our database of abstract order types: It is valid for $n \leq 8$ points, but for $n = 9$ points the list of **15296266** abstract order types contains one counter-example, and this is the only one for $n = 9$. Moreover this abstract order type has been found to be realizable; a picture of the counter-example is given in Figure 5.

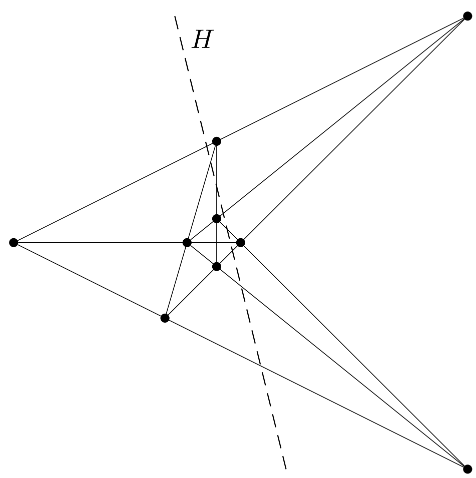

Figure 5: Da Silva-Fukuda–Conjecture: The Counter-Example with 9 Points

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