Tutte's Barycenter Method applied to Isotopies

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Abstract

This paper provides a simple proof of Tutte's barycentric embedding theorem [38]; a counterexample showing that Tutte's theorem does not hold in dimensions higher than 3; and the description and analysis of a method to build isotopies of triangulations in the plane, based on Tutte's theorem and the computation of equilibrium stresses of graphs by Maxwell-Cremona's theorem.

1 Introduction

Tutte's theorem. In 1963, Tutte [38] gave a way to build embeddings of any planar, 3-connected graph $G = (V, E)$. Let C be a cycle whose vertices are the vertices of a face of G in some (not necessarily straight-line) embedding Γ' of G. Let Γ be a straight-line mapping of G into the plane, satisfying the conditions:

i. the set V_e of the vertices of the cycle C is mapped to the vertices of a strictly convex polygon Q, in such a way that the order of the points is respected;

ii. each vertex in $V_i = V \setminus V_e$ is a barycenter with positive coefficients of its adjacent vertices (Tutte assumed all coefficients to be equal to 1, but the proof extends without changes to this case). In other words, the images $\overline{v}$ of the vertices v under Γ are the solution of a linear system (S): for each $u \in V_i$, $\sum_{v|uv\in E} \lambda_{uv}(\overline{u} - \overline{v}) = 0$, where the λ_{uv} are positive reals. It can be shown that the system (S) admits a unique solution.

Theorem 1 (Tutte's Theorem) *Γ is an embedding of G into the plane, with strictly convex interior faces.*

Several theorems for embedding graphs exist in the literature. Early works by Fáry [18] (resp. Stein [33]) show that any planar graph admits a straight-line (resp. straight-line, with convex faces under some conditions) embedding. More recent works focus on finding embeddings of graphs so that the coordinates of the vertices are integers with absolute value as small as possible; there is an algorithm [29] to embed graphs with $n + 2$ vertices on the $n \times n$ grid. Other criteria are also desirable, such as the convexity of the faces and/or minimizing the area of the embedding if a minimum distance between two vertices, or between a vertex and a non-incident edge, is imposed [10]; Tutte's method with unit coefficients can yield embeddings with exponential area under such a restriction [15]. A topic of interest is also to realize 3-connected, planar graphs as the 1-skeleton of a 3D polytope: this can be done on the cubic grid of size $2n^{169n^3}$ [27]. For a survey on algorithms for graph drawing, see [14]. Tutte's method is often used in graphics applications related to surface parametrization in multiresolution problems [16] and geometric modeling [19], texture mapping [25], and morphing [23, 20, 21].

In his paper [38], in addition to showing Theorem 1, Tutte simultaneously proves again Kuratowski's planarity criterion [24] of 1930: a graph is planar unless it contains a subdivision of one of the two Kuratowski graphs K_5 and $K_{3,3}$. The proofs of both results are entangled together in Tutte's paper; the consequence is that proving Theorem 1 by his method is long and involves quite a lot of graph theory terminology. Later, short proofs of Kuratowski's criterion were given by Thomassen [35],

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making Tutte's graph-theoretic viewpoint less attractive and unnecessarily complicated if the goal is to show Theorem 1. As we will see, starting with the topological fact that C is a facial cycle of some embedding of G and using Euler's Formula yields a readable, short proof.

Other proofs of this theorem exist in the literature, using a more geometric viewpoint, but they are not completely satisfying. Becker and Hotz [2] use the notion of "quasi-planarity" as limit case of a planar situation, which yields to complicated notations and tedious case analyses; the structure of their paper is non-obvious and the proof is really long. Y. Colin de Verdière [13] shows the result, only for triangulated graphs, on arbitrary surfaces of non-positive curvature using the Gauss-Bonnet formula; the same technique does not seem to extend to 3-connected graphs [12].

In the same spirit as Thomassen [34, 35, 36], who gave simple proofs of basic theorems in graph theory, we provide a short, self-contained proof of Tutte's theorem. Its attractive points are its shortness and its simplicity (very few graph theory is required, the geometric aspects are emphasized). The proof is transparent and progressive; it consists of two clearly delimited stages. First, we show Tutte's result without effort under two additional restrictions:

iii. the graph G is triangulated: every face of Γ', except possibly the face corresponding to the cycle C, is a triangle;

iv. the images of these triangles under the mapping Γ are non-degenerate, *i.e.*, their interior is non-empty.

After that, we deal with the degeneracies, which are the core of the problem (in our paper as well as in other proofs): we show that under hypotheses (i) and (ii), three vertices belonging to a face are not on the same line. This step uses the non-planarity of $K_{3,3}$ together with simple geometric ideas. Then, the generalization to arbitrary 3-connected graphs comes easily.

Isotopies. Tutte's theorem yields a method, described by Floater and Gotsman [20] and Gotsman and Surazhsky [21], to morph two triangulations, the boundary being the same convex polygon in both embeddings. One can compute coefficients $\lambda_{uv} > 0$, for each interior vertex u and each neighbor v of u, so that u is the barycenter with coefficients $(\lambda_{uv})_v$ of its neighbors in the initial embedding. Doing the same for the final embedding and interpolating linearly the coefficients yields an isotopy (a continuous family of embeddings) by Tutte's theorem. This method lets some freedom for the computation of the barycentric coefficients of the vertices in both embeddings. Hence, we study the following natural question: is it possible to apply the same technique, with the additional restriction that the coefficients are symmetric ($\lambda_{uv} = \lambda_{vu}$)? The interest is that this has a clear and appealing physical interpretation: fix the exterior vertices and edges and replace each interior edge joining two vertices u and v by a spring with rigidity λ_{uv}; then the equilibrium state of this physical system is the solution of the system (S). The problem of computing such symmetric coefficients is solved with Maxwell-Cremona's theorem from rigidity theory. The drawback of our method is that these coefficients are not always positive, hence Tutte's theorem does not apply in all cases. After small experiments (with 20 vertices or so), we thought that our method always yielded an isotopy, even if some weights were negative. This is not the case, and we have small examples refuting this conjecture. However, our method gives positive coefficients if both embeddings are in the rather general class of regular triangulations (recall that a regular subdivision is the projection of the lower faces of a polytope generated by a family of points, see [42]). This idea of replacing edges of a graph by springs has been used in several other contexts: in mechanics [40], for graph connectivity computation [26], in an algorithmic study of operations on polyhedra [22].

Generalization to 3D space. The last part of this paper is devoted to the study of the extension of Tutte's theorem to three dimensions. It presents an overview of the proof that there exist two triangulations of a tetrahedron which are combinatorially equivalent but for which there is yet no linear isotopy from one to the other, a fact which is specific to spaces of dimension ≥ 3. This result has been shown by Starbird in [30]; we give an outline of the proof and explain parts of the proof not written in his paper and required to show this theorem. Then we show that the natural generalization of Tutte's barycentric embedding theorem is false in 3D. The translation of Tutte's hypotheses (in the triangulated case) from 2D to 3D is as follows: consider an embedding of a simplicial 3-complex K into $\mathbb{R}^3$, the boundary being a convex polyhedron. If a mapping of K into $\mathbb{R}^3$, with the same boundary, is so that each interior vertex is barycenter with positive coefficients of its neighbors, then we would expect that it is an embedding. It turns out that this fact is false. To our knowledge, this attempt of generalizing Tutte's theorem for 3D complexes is new, and our refutation of this extension raises interesting open questions, in the context of isotopies as well as in view of embedding 3-complexes.

2 Proof of Tutte's theorem

We prove here Tutte's theorem ([38], relying on [37]). For basic graph theory definitions, see for example [6]. A *mapping* Γ of G into the plane is an application $\Gamma : V \cup E \to \mathcal{P}(\mathbb{R}^2)$ which maps a vertex $v \in V$ to a point in $\mathbb{R}^2$ and an edge $e = uv \in E$ to the straight line segment joining $\Gamma(u)$ and $\Gamma(v)$. An *embedding* Γ of G is a mapping of G so that distinct vertices are mapped to distinct points, and the images of distinct edges can only meet at their endpoints.

We first rephrase condition (ii) in more compact terms. Define the *strict convex hull* of a set of points to be the interior, in the space affinely generated by these points, of the convex hull of these points. It is then easy to see that condition (ii) is equivalent to the following:

ii'. each $v \in V_i$ lies in the strict convex hull of its adjacent vertices.

A proof of this equivalence is provided for completeness in Appendix A. We thus need not use System (S) anymore; the proof of its invertibility is easy and not necessary for the proof of Tutte's theorem, we defer it to Appendix B.

Recall that Γ' is a (not necessarily straight-line) embedding of G with facial cycle C. In the sequel, Γ' is our reference embedding, and we shall call the *faces of* G (or *triangles* if (iii) is satisfied) the faces of the embedding Γ' except the face bounded by C. We first state a preliminary lemma, assuming the hypotheses of Theorem 1:

Lemma 2 *Each $v \in V_i$ is mapped by Γ into the interior of the polygon Q.*

Proof. First, each vertex in V_i is mapped, by Γ, into the interior or the boundary of the polygon Q. For if this is not the case, by convexity of Q, there is a vertex $v \in V_i$ so that Q and $\overline{v}$ are separated by a line D. Among the vertices whose images under Γ lie on the same side of D as v, consider those which are the farthest from D. Obviously, at least one of these vertices cannot be in the strict convex hull of its adjacent vertices.

Suppose that a vertex $v \in V_i$ is mapped into the boundary of Q, on a line D which contains an edge of Q. Because the image of any vertex lies in the same (closed) half-plane bounded by D, and by condition (ii'), the vertices in V_i which are adjacent to v are also contained in D. Thus, all vertices of the connected component of v in $G - V_e$ lie on D. This contradicts the 3-connectivity of G, because removing the two vertices of V_e which are on D destroys the connectivity of G. $\square$

Our intermediate goal is now to prove the theorem under additional assumptions (iii) and (iv). We first show a local planarity property for Γ.

Lemma 3 *Under assumptions (iii) and (iv), the interiors of the images of two distinct triangles of G which share a common vertex do not overlap.*

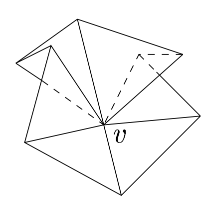

Figure 1: The triangles with v as a vertex, involved in the computation of $\sigma(v)$.

Proof. By (iv), the angles of a triangle are well-defined and positive. We first introduce some terminology. For each vertex v, let $\alpha(v)$ be equal to 2π if $v \in V_i$, and to the angle of the polygon Q at v if $v \in V_e$; let $\sigma(v)$ be the sum, over all the triangles incident to v, of the angle of the image of such a triangle at v under the mapping Γ (Figure 1). Our aim is to show that, for each vertex v,

$\sigma(v) = \alpha(v)$, that is, there is no "folding" at v in the mapping Γ. Because all faces of G are triangles, the structure of G in the neighborhood of a vertex v is quite simple: the vertices adjacent to v form a cycle (if $v \in V_i$) or a path (if $v \in V_e$). Let us call $v_1, \ldots, v_p$ the neighbors of v in the order of this cycle (or path).

A key ingredient of the proof of Lemma 3 is the following fact: $\sigma(v) \geq \alpha(v)$, with equality if and only if the triangles incident to v do not overlap. To prove this, let $\theta(uvw)$ be the geometric angle (between 0 and π) of the triangle uvw at v in the mapping Γ, and assume v to be in V_i (the proof is easier for v in V_e). By (ii') and (iv), $\overline{v}$ lies in the interior of the convex hull of $\overline{v}_1, \ldots, \overline{v}_p$. It is then easy to see that there exist i, j and k so that $1 \leq i < j < k \leq p$ and that the sum of the angles $\theta(v_i v v_j)$, $\theta(v_j v v_k)$ and $\theta(v_k v v_i)$ equals 2π. We have $\sum_{q=i}^{j-1} \theta(v_q v v_{q+1}) \geq \theta(v_i v v_j)$, and similar relations between j and k and between k and i. Adding these three inequalities, we obtain that $\sigma(v) \geq 2\pi$. Moreover, equality holds if and only if there is equality in all previous sums, that is, the ordering of the vertices $v_1, \ldots, v_p$ around v is preserved in Γ, which is also equivalent to the fact that the triangles incident to v do not overlap.

Now, let t be the total number of triangles. We have:

$$(2|V_i|+|V_e|-2)\pi = \sum_{v \in V} \alpha(v) \leq \sum_{v \in V} \sigma(v) = \pi t. \quad (1)$$

The first equality is a consequence of the fact that the sum of the angles of the polygon Q is $(|V_e| - 2)\pi$, the inequality has been shown above, and the second equality is true because the sum of the angles of a triangle in the plane equals π. But on the other hand, the leftmost and rightmost members of (1) are equal. Indeed, Euler's formula, applied to the planar graph G, yields (if e is the number of interior edges):

$$(|V_i| + |V_e|) - (|V_e| + e) + (t + 1) = 2. \quad (2)$$

The fact that every face of G has three edges is expressed by:

$$3t = 2e + |V_e|. \quad (3)$$

Combining equations (2) and (3) to eliminate e leads to equality of the extreme members of (1), as claimed. Since also $\sigma(v) \geq \alpha(v)$, we get: for each $v \in V, \sigma(v) = \alpha(v)$. Using then the equality case in the fact given above, we obtain that the triangles incident to v do not overlap, which concludes the proof. $\square$

We now show global planarity, that is, Γ is an embedding.

Lemma 4 *Under restricting assumptions (iii) and (iv), Theorem 1 holds.*

Proof. First note that the vertices, edges and triangles of G define an (abstract) 2-dimensional simplicial complex K. In fact, one can view K as a 2-manifold with boundary: each point of this manifold is defined by its barycentric coordinates in a triangle of G, with the obvious identifications of points on edges or vertices. Γ induces a map from this manifold K into Q: for each point p in K, determined by its barycentric coordinates in a triangle uvw, its image $\Gamma(p)$ is the point in the triangle $\overline{uvw}$ with the same barycentric coordinates. In this setting, Γ is clearly continuous and even a local homeomorphism: each point p has a neighborhood $N(p)$ so that $\Gamma|_{N(p)}$ is a homeomorphism on its image set. This is clear for points p in the interior of a triangle and Lemma 3 proves this fact if p belongs to an edge or is a vertex of a triangle. For $a \in Q$, $n(a) = |\Gamma^{-1}(a)|$ is finite; for otherwise, by compactness, there would be an accumulation point of the set $\Gamma^{-1}(a)$, contradicting the local homeomorphism property.

Let $a \in Q$ and $p = n(a) \geq 0$; we show that $n(b) = p$ for b sufficiently close to a (that is, the function n is locally constant). Let $N_1, \ldots, N_p$ be disjoint open neighborhoods of each of the points in $\Gamma^{-1}(a)$, chosen small enough so that $\Gamma|_{N_i}$ is a homeomorphism for each i. Let $N = \Gamma(N_1) \cap \cdots \cap \Gamma(N_p)$ is a neighborhood of a; $F = \Gamma(K \setminus (N_1 \cup \cdots \cup N_p))$ is a compact set which does not contain a; hence $N' = N \setminus F$ is a neighborhood of a. Each $b \in N'$ has exactly p preimages in $N_1 \cup \cdots \cup N_p$ because $b \in N$ and no preimage outside this set because $b \notin F$. Thus, by connectivity of Q, n is constant; its value is 1 on the boundary of Q by Lemma 2, hence Γ is a homeomorphism. The proof is complete. $\square$

This proves the theorem in a particular case; we will use this result in the sequel. From this point, unless stated otherwise, we do not assume conditions (iii) and (iv) anymore, but only the hypotheses of Theorem 1. The goal is to show that some degenerate cases cannot occur, using the 3-connectivity of G. We first state a quite general lemma, inspired by Tutte [38], which we call the Y–lemma in view of the geometry of the problem. The situation is depicted in Figure 2. Note that, in this paper, any path in a graph is supposed to be simple and non-degenerate.

Lemma 5 (Y–lemma) *Let v_1, v_2, v_3 and v be pairwise distinct vertices of a graph H. Assume, for $i = 1, 2, 3$, that there is a path P_i from v_i to v which avoids the v_j's (for $j \neq i$). Then there exist three paths P'_i, from v_i to a common vertex v', which are pairwise disjoint (except at v').*

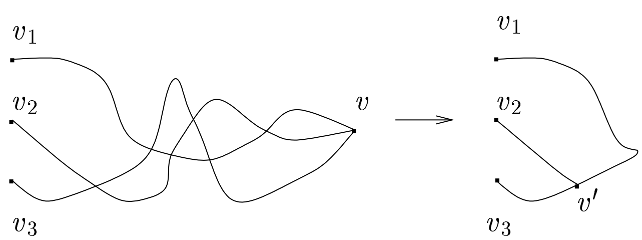

Figure 2: The situation in the Y–lemma.

Proof. First, using P_1 and P_2, we easily get a (simple) path R from v_1 to v_2, so that R and P_1 have the same first edge $v_1 z$. Then we consider the path P_3. If this path P_3 intersects R, let v' be the first vertex of intersection on P_3. v' splits R in two parts, which we call P'_1 (from v_1 to v') and P'_2 (from v_2 to v'); P'_3 is the part of P_3 going from v_3 to v', with loops removed (if any). The P'_i's satisfy the property stated in the lemma. If P_3 does not intersect R, we call v' the last vertex on P_1 (when going from v_1 to v) which is also on R. Such a vertex exists and is different from v_1 because $v_1 z$ is the first edge of R and P_1. Let P'_3 be the path defined by P_3 followed by the part of the path P_1 which goes from v to v', with loops removed (if any). v' splits R in two parts, which we call P'_1 and P'_2. The paths P'_i's satisfy the desired property. $\square$

We now come back to the situation of Theorem 1, and we introduce some geometric definitions, partially taken from [38]. We will represent a line in the plane by the zero set of a non-constant affine form. Henceforth, φ is such an affine form. A vertex v of G is called *φ-active* if there is a vertex v' adjacent to v so that $\varphi(\overline{v}) \neq \varphi(\overline{v}')$, *$\varphi$-inactive* otherwise. The *φ_+-poles* are the vertices $v \in V$ so that $\varphi(\overline{v})$ is maximal; the definition for the *φ_--poles* is analogous. The *φ-poles* are the φ_+-poles and the φ_--poles. By Lemma 2, a φ-pole must be in V_e. It is then clear that there are exactly one or two φ_+-poles and that, in the latter case, they are connected by an edge of Q. If $\overline{v}_1, \ldots, \overline{v}_k$ lie on the line $\varphi = 0$, $G(\varphi_+, v_1, \ldots, v_k)$ is the graph induced by the vertices lying in the half-plane $\varphi > 0$, to which we add the vertices $v_1, \ldots, v_k$ and all edges from one of these vertices to a vertex in $\varphi > 0$. Let $G(\varphi)$ be the subgraph of G induced by the vertices v lying on the line $\varphi = 0$. The following lemma was also shown in [38].

Lemma 6 *Let v be a φ-active vertex so that $\varphi(\overline{v}) = 0$; assume that v is not a φ_+-pole. Then there exists a path in $G(\varphi_+, v)$ from v to a φ_+-pole of G.*

Proof. The problem boils down to this: given a φ-active vertex w, which is not a φ_+-pole, prove that it is possible to find a neighbor of w which has a greater value of φ and is also φ-active. If $w \in V_e$, clearly, there exists in V_e a vertex adjacent to w which has a greater value of φ; this vertex is also φ-active. If $w \in V_i$, then w has neighbors in both increasing and decreasing directions of φ, because a vertex is in the strict convex hull of its adjacent vertices (hypothesis (ii')) and because w is φ-active. It is therefore possible to find an adjacent vertex with a greater value of φ. This vertex is also φ-active. $\square$

The two following lemmas show that some degenerate cases cannot occur. The first one is along the lines of [38], contrary to the second one which uses another argument.

Lemma 7 *For any φ, G has no φ-inactive vertex.*

Proof. Suppose that there is a φ-inactive vertex v. Figure 3 summarizes the proof: we show that the planar graph G contains a subdivision of the bipartite graph $K_{3,3}$, which is impossible

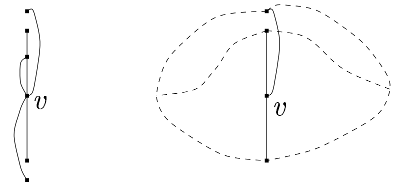

Figure 3: A summary of the proof of Lemma 7.

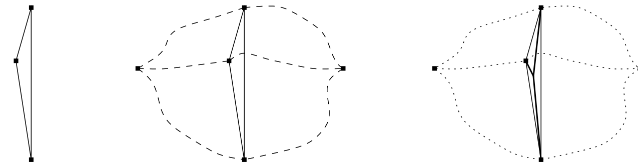

Figure 4: A summary of the proof of Lemma 8.

(see for example [36]). Using the fact that G is 3-connected, we can see the existence, in $G(\varphi)$, of three distinct φ-active vertices $v_i, i = 1, 2, 3$, and three paths P_i joining v to v_i, so that, for any i, the path P_i does not contain any vertex v_j for $j \neq i$. Indeed, let w be a vertex of G so that $\varphi(\overline{w}) \neq 0$. By connectivity of G, take a path from v to w and, on this path, take the first φ-active vertex and call it v_1. Do the same in $G - \{v_1\}$ and choose v_2 (use 2-connectivity). Finally, use 3-connectivity to select v_3 in $G - \{v_1, v_2\}$.

Applying then the Y–lemma in $G(\varphi)$, we get the existence of a vertex v' in $G(\varphi)$, together with three distinct paths (except at v') P_i from v_i to v' in $G(\varphi)$. We now use Lemma 6. We have the existence, in $G(\varphi_+, v_1, v_2, v_3)$, of three paths Q_i joining v_i to a vertex x so that $\varphi(\overline{x}) > 0$. Then, the Y–lemma allows us to assume, by changing x and the Q_i's if necessary, that these three paths are disjoint (except at x). Similarly, in $G(\varphi_-, v_1, v_2, v_3)$, we have three disjoint paths R_i joining v_i to a vertex y so that $\varphi(\overline{y}) < 0$. Using the paths P_i, Q_i and R_i, which are all pairwise disjoint except at their endpoints, and the vertices x, v', y and v_1, v_2, v_3, we get a subdivision of the graph $K_{3,3}$. This contradicts the planarity of G. □

Lemma 8 *Let v_i, $i = 1, 2, 3$, be three vertices of a face of G. Then, under Γ, the v_i's are not collinear.*

Proof. Suppose the v_i's are on the line $\varphi = 0$. Figure 4 gives the essential ideas of the proof: we again find a subdivision of $K_{3,3}$. By Lemma 7, the v_i's are φ-active. Since the v_i's are collinear, at least one of them is in V_i, so none of them is a φ-pole. Again, Lemma 6 and the Y–lemma show the existence of a vertex x and three disjoint paths (except at x) Q_i joining v_i to x in $G(\varphi_+, v_1, v_2, v_3)$. Using a similar argument on the other side of the line $\varphi = 0$, we finally obtain the existence of x, y and six disjoint (except at their endpoints) paths joining x or y to the v_i's. Let G' be the graph G to which we add a vertex w linked to the v_i's. Because the v_i's belong to a common face, G' is planar. But it also contains a subdivision of $K_{3,3}$ (with the six paths described above and the three new paths joining w to the v_i's), which is impossible. □

The two previous lemmas have been shown merely under the hypotheses of Theorem 1. Lemma 7 will be used to deal with the non-triangulated case. But here, as a straightforward consequence, we get the proof of the theorem in the triangulated case:

Corollary 9 *Theorem 1 holds when (iii) is satisfied, that is, in the particular case where each face of G is a triangle.*

Proof. Indeed, assuming condition (iii), Lemma 8 states that condition (iv) is also true. We can thus apply Lemma 3. Therefore, Γ is an embedding. □

We can now prove Theorem 1 in full generality. We first triangulate G. More precisely, this means that edges are added to split the faces of G in triangles, without adding vertices (this is done in a purely combinatorial way: no geometry is involved here). Let G_1 be this planar augmented graph. Adding the same edges in the mapping Γ gives us a mapping Γ_1 of the graph G_1. We now check that we can apply Corollary 9 to G_1 and Γ_1. In fact, this boils down to checking condition (ii') to Γ_1. By condition (ii') and Lemma 7 applied to Γ, the neighbors of an interior vertex are not all on a line (under Γ), and such a vertex is in the interior of the convex hull of its neighbors. Because Γ_1 is obtained from Γ by adding extra edges, condition (ii') also holds for Γ_1. Thus, by Corollary 9, Γ_1 is an embedding. Deleting the edges we added earlier to Γ, we obtain that Γ is an embedding as well. It is clear that the faces are strictly convex.

3 Isotopies in the plane

Now, we detail the construction of the isotopy outlined in the introduction. Let $G = (V, E)$ be a 3-connected planar graph and let Γ_0 and Γ_1 be two embeddings of G into the plane. We look for an isotopy between Γ_0 and Γ_1, restricting ourselves to the following situation: the boundary cycle C of the exterior face of Γ_0 is a convex polygon, it bounds also the exterior face of Γ_1, and the corresponding vertices of C are at the same location in Γ_0 and Γ_1. During the isotopy, the vertices of C have to remain at the same position. In addition, we will require the graph G to be triangulated.

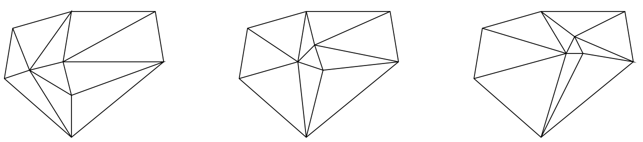

Figure 5: An isotopy Γ_t ($t \in [0, 1]$) in our framework: here Γ_0, $\Gamma_{1/2}$ and Γ_1 are depicted.

A natural idea arising to solve this problem is the following: try to deform Γ_0 into Γ_1 by keeping the exterior vertices at the same place and moving the interior vertices linearly. That is, $\Gamma_t(v) = (1-t)\Gamma_0(v)+t\Gamma_1(v)$ for an interior vertex v and t in $[0, 1]$. It turns out that this approach does not always yield an isotopy, as Figure 6 demonstrates. Bing and Starbird [4], generalizing a result by Cairns [8], showed the existence of an isotopy in the context described above. However, these papers do not provide an algorithmic solution to this problem.

As explained in introduction, Gotsman et al. [20, 21] gave a method, based on Tutte's theorem, to solve this isotopy problem, representing a vertex as barycenter of its neighbors. We will use the following definitions in order to study the case where the barycentric coefficients are symmetric. Let E_i be the set of (undirected) interior edges (the edges for which at least one incident vertex is in V_i). A *weight function* on Γ, or *stress*, is a map $\omega : E_i \to \mathbb{R}$; hence $\omega_{uv} = \omega_{vu}$. ω is *positive* if $\omega_{uv} > 0$ for each interior edge uv. If ω and the positions of each $v \in V_e$ are fixed, the *equilibrium state* is defined by the system: for each $u \in V_i, \sum_{v|uv\in E} \omega_{uv}(\overline{u} - \overline{v}) = 0$. In these conditions, ω is an *equilibrium stress* for Γ.

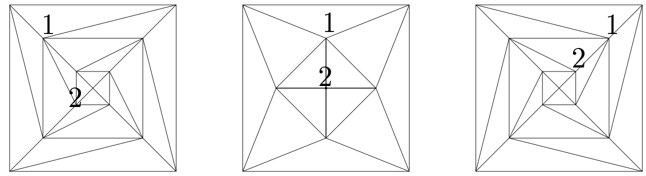

Figure 6: An example showing that the naive approach does not work. The figure shows Γ_0 (left) and Γ_1 (right). The two inner squares are "twisted" to the left (resp. right) under Γ_0 (resp. Γ_1), and the innermost square must rotate by an angle of π in the whole motion. With the linear motion, the vertices of the inner square would collapse at $t = 1/2$, as shown in the picture in the middle. Therefore, this motion does not yield an isotopy.

Here is a summary of our approach: compute equilibrium stresses ω^0 (resp. ω^1) of embeddings Γ_0 (resp. Γ_1); then, for $t \in [0, 1]$, compute the equilibrium state of $\omega^t = (1-t)\omega^0 + t\omega^1$. The difficulty resides in computing an equilibrium stress for a given embedding Γ: our method relies on the Maxwell-Cremona correspondence, a theorem well-known in rigidity theory (see Hopcroft and Kahn [22] for details). Think of Γ as being in the plane $z = 0$ of $\mathbb{R}^3$. Take any *lift* of Γ, by adding to each vertex $\bar{v} = p_v = (x_v, y_v, 0)$ of Γ a third coordinate, leading to $q_v = (x_v, y_v, z_v)$. Consider the polyhedral terrain whose vertices are the q_i's and which has the same incidence structure as Γ (Figure 7). Now, let ij be an interior edge of Γ; let l and r be the left and right neighbor of the (oriented) edge ij (Figure 8) and φ^L_{ij} (resp. φ^R_{ij}) the affine form which takes the value z_i, z_j, z_l (resp. z_r) at points p_i, p_j, p_l (resp. p_r). We will define an equilibrium stress for Γ determined by this lift.

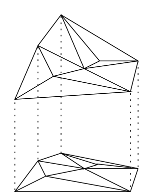

Figure 7: A lift of an embedding.

If $a_0, \dots, a_k$ are $k + 1$ points of $\mathbb{R}^k$, written as column vectors, we introduce the multi-affine bracket operator $[a_0, \dots, a_k]$, defined by

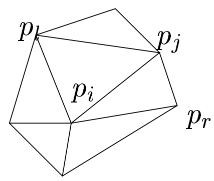

Figure 8: The notations for the computation of ω_{ij}.

$[a_0, \ldots, a_k] = \begin{vmatrix} a_0 & a_1 & \ldots & a_k \\ 1 & 1 & \ldots & 1 \end{vmatrix}$ (this quantity being proportional to the signed volume of the convex hull of the a_i's).

Lemma 10 *For each interior edge ij and any $p \in \mathbb{R}^2$,*

$$\varphi^L_{ij}(p) - \varphi^R_{ij}(p) = \frac{[p_i, p_j, p]}{[p_i, p_j, p_l]}(\varphi^L_{ij}(p_l) - \varphi^R_{ij}(p_l)).$$

Proof. It is a consequence of Cramer's formula. Let φ be an affine form on $\mathbb{R}^k$ and $a_0, \ldots, a_k$ be $k+1$ affinely independent points, $a \in \mathbb{R}^k$. Let $\alpha_0, \ldots, \alpha_k$ be the barycentric coordinates of a with respect to the a_i's, that is, by definition:

$$\begin{array}{ccccccc} \alpha_0 a_0 & + & \ldots & + & \alpha_k a_k & = a \\ \alpha_0 & + & \ldots & + & \alpha_k & = 1. \end{array}$$

Cramer's formula now implies:

$$\alpha_i = \frac{[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_k]}{[a_0, \ldots, a_k]}.$$

So (if $k = 2$, and because φ is an affine form): $\varphi(a) = \frac{[a, a_1, a_2]}{[a_0, a_1, a_2]}\varphi(a_0) + \frac{[a_0, a, a_2]}{[a_0, a_1, a_2]}\varphi(a_1) + \frac{[a_0, a_1, a]}{[a_0, a_1, a_2]}\varphi(a_2)$. It is now easy to conclude. $\Box$

Define, for any interior edge ij and for a point p not on the line $(p_i p_j)$:

$$\omega_{ij} = \frac{\varphi^L_{ij}(p) - \varphi^R_{ij}(p)}{[p_i, p_j, p]}.$$

This definition does not depend on the point p, by Lemma 10. Furthermore, $\omega_{ij} = \omega_{ji}$. In practice, there is an intrinsic formula (recall that the q_i's are the lifts of the points p_i's, which are the images of the vertices under Γ):

Lemma 11 $\omega_{ij} = \dfrac{[q_i, q_j, q_l, q_r]}{[p_i, p_j, p_l][p_i, p_j, p_r]}.$

Proof. By definition of ω_{ij}:

$$\omega_{ij}[p_i, p_j, p_l][p_i, p_j, p_r] = (z_l - \varphi^R_{ij}(p_l))[p_i, p_j, p_r]. \tag{4}$$

By Cramer's formula, as in the proof of Lemma 10: $\varphi^R_{ij}(p_l)[p_i, p_j, p_r] = z_i[p_l, p_j, p_r] + z_j[p_i, p_l, p_r] + z_r[p_i, p_j, p_l]$. Thus the left member of Equation (4) equals

$$z_l[p_i, p_j, p_r] - z_i[p_l, p_j, p_r] - z_j[p_i, p_l, p_r] - z_r[p_i, p_j, p_l],$$

which equals $[q_i, q_j, q_l, q_r]$ (by developping this determinant with respect to the third line). $\Box$

Theorem 12 *ω is an equilibrium stress for Γ.*

Proof. For any point p in the plane, $i \in V_i$, we have: $\sum_{j|ij \in E} \omega_{ij}[p_i, p_j, p] = \sum_{j|ij \in E}(\varphi^L_{ij}(p) - \varphi^R_{ij}(p)) = 0$, because the affine form φ corresponding to a face incident to p_i appears twice in this sum, once counted positively, once negatively. As $[p_i, p_j, p] = \det(p_j - p_i, p - p_i)$, this implies $\det(\sum_{j|ij \in E} \omega_{ij}(p_i - p_j), p - p_i) = 0$, for each point p in $\mathbb{R}^2$. Therefore $\sum_{j|ij \in E} \omega_{ij}(p_i - p_j) = 0$. $\Box$

Thus, each lift of the embedding Γ determines an equilibrium stress on Γ. Conversely, it is possible to show that an equilibrium stress determines a unique lift of Γ, up to the choice of an affine form of $\mathbb{R}^2$ (Maxwell's theorem, shown for example in [22] in a slightly different context).

If we have *positive* equilibrium stresses ω^0 and ω^1 of Γ_0 and Γ_1 respectively, we have a method to compute an isotopy between Γ_0 and Γ_1: by Tutte's theorem, because $\omega^t = (1-t)\omega^0 + t\omega^1$ is a positive stress for each $t \in [0, 1]$, the corresponding mapping Γ_t is an embedding, and $(\Gamma_t)_{t \in [0,1]}$ is clearly continuous (the map which associates to each invertible matrix its inverse, is continuous), hence an isotopy. Furthermore, it is easy to characterize the set of embeddings which admit a positive equilibrium stress: an edge ij has a positive weight if and only if the line $q_i q_j$ (with the notations above) is under the line $q_l q_r$; hence an embedding has a positive stress if and only if it is a regular triangulation. Therefore, we have:

Theorem 13 *If Γ_0 and Γ_1 are regular triangulations, then we can compute an isotopy between Γ_0 and Γ_1.*

Testing whether Γ is a regular subdivision, and, if so, computing a positive lift, can be done easily using linear programming; indeed, we have a convex lift for Γ if and only if, for each interior edge ij and with the notations above, $[q_i, q_j, q_l, q_r] < 0$, which is a linear inequality in the z_k's. Not all triangulations are regular subdivisions, as shown in Figure 9 (see [42, p. 132]), but a large class of embeddings are regular subdivisions, including Delaunay triangulations for example (because the Delaunay triangulation of a set of points is the projection of the edges of the convex hull of the points lifted on the standard paraboloid, see [5, p. 437] or [17]); this remark might be useful because of the wide use of these triangulations in computational geometry.

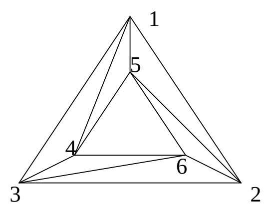

Figure 9: An embedding which is not a regular subdivision. Indeed, assuming it is possible to lift it to a lower convex hull, we can suppose, by adding a suitable affine form to all the z_i's, that $z_4 = z_5 = z_6 = 0$. If this graph were a regular subdivision, we would have $z_1 > z_2 > z_3$, which is impossible.

Remark. We only studied triangulated graphs in this section because it is probably easier to deal with them than with general planar 3-connected graphs. However, the same theory applies if the graph is only 3-connected. The definition of a lift must be adapted: all the vertices belonging to the same face must be lifted on a common plane in 3D space (it also corresponds to triangulating the graph and putting a weight equal to zero on these new edges); testing whether we have a regular subdivision is also a linear programming problem.

In practice, we tried to build an isotopy between a random triangulated embedding and the "canonical" embedding of the same graph (that is, the embedding obtained by Tutte's method when all weights equal 1). We lift Γ_0 to the standard paraboloid $z = x^2 + y^2$, compute the equilibrium stress, and use linear interpolation between ω^0 and ω^1. Although the initial stress is not necessarily positive, it turns out that, in many (not too big) cases, this method yields an isotopy; long experiments have been necessary to find a small counterexample like Figure 10. Our smallest counterexample uses 4 outer vertices and 2 inner vertices, but the failure is very hard to see on the screen and can only be proved by computation. Lifting on the paraboloid may give an isotopy even if the considered triangulation is non-regular, like in Figure 6, but can also fail with regular triangulations (the initial and final triangulations in Figure 10 are regular). This method has been programmed in C++ using Numerical Recipes and the LEDA library, and also in Mathematica for exact computations. Coordinates of the examples are available at [1].

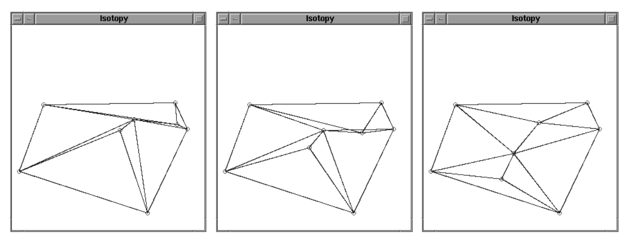

Figure 10: An example of non-planarity with the lift on the standard paraboloid.

Several other approaches could be done in the same spirit to try to find a method which would work for a larger class of embeddings than the regular subdivisions. One could attempt to study the space of stresses which yield an embedding (thus an isotopy corresponds to a path in this space). If we restrict ourselves to the linear interpolation between the weights, an important question is: are there two embeddings Γ_0 and Γ_1 so that, for any lifts of Γ_0 and Γ_1, the interpolation $\omega^t = (1-t)\omega^0 + t\omega^1$ of the corresponding weights does not yield an isotopy? If it is not the case, how to compute the lifts?

Finally, in our attempts to disprove the conjecture stating that a lift on the standard paraboloid always yields an embedding, we noticed that the matrix involved in the computation of the positions of the points was symmetric positive definite in all our experiments; whether this result is always true remains an open question. If it is the case, it has the following interesting consequence. If ω is a stress on G, let us denote by M_ω the matrix involved in the inversion of System (S). It can be shown (see the proofs of Lemma 18) that M_ω is symmetric positive definite if ω is positive; moreover, $\omega \mapsto M_\omega$ is linear. If M_{ω^0} and M_{ω^1} are symmetric positive definite, so is $M_{(1-t)\omega^0+t\omega^1} = (1-t)M_{\omega^0} + tM_{\omega^1}$, and uniqueness of the positions of the vertices is guaranteed during the motion (which may fail to be an isotopy). Similarly, if M_{ω^0} is symmetric positive definite and ω^1 is a positive stress, since multiplying ω^1 by a positive number does not affect the equilibrium state, we can assume $\omega^1 \geq \omega^0$ (this notation simply means that for each interior edge ij, $\omega^1_{ij} \geq \omega^0_{ij}$). Each nondecreasing family ω^t of stresses from ω^0 to ω^1 yields a family M_{ω^t} of symmetric positive definite matrices; indeed, $M_{\omega^t} = M_{\omega^0} + M_{\omega^t - \omega^0}$; the first matrix of the right term is symmetric positive definite, the second one is positive because the corresponding stress is non-negative on each interior edge. Thus, if this conjecture is true, the positions of the vertices are uniquely determined for many choices of the interpolation between the weights.

4 Generalization to 3D space

We explain here why the analogue of Tutte's theorem is false in 3D space, thus making it difficult to build isotopies in 3D. The vocabulary of this section is different from the one used in the proof of Tutte's theorem; it is more convenient to use combinatorial simplicial complexes (all simplicial complexes considered here are combinatorial, not geometric; see for example [39]).

We introduce some other definitions, generalizing those in 2D. A *mapping* f from a simplicial complex C into $\mathbb{R}^d$ is a map from all the simplexes of C into $\mathcal{P}(\mathbb{R}^d)$ satisfying: if $\{v_1, \ldots, v_p\}$ is a simplex of C, $f(\{v_1, \ldots, v_p\}) = \mathrm{Conv}\{f(v_1), \ldots, f(v_p)\}$. An *embedding* of C into $\mathbb{R}^d$ is a mapping so that, for any two simplexes $\sigma, \tau \in C$, $f(\sigma \cap \tau) = f(\sigma) \cap f(\tau)$. As usual, an *isotopy* $(h(t))$ $(t \in [0,1])$ of C into $\mathbb{R}^d$ is a continuous family of embeddings of C into $\mathbb{R}^d$. Finally, the *image* of a simplicial complex C by a mapping f is the union of the sets $f(\tau)$, over all simplexes τ of C.

In this section, we will often manipulate complexes whose embeddings have to be fixed on the "boundary" of these complexes. A *3-complex with tetrahedral boundary* (C, B, b) is a simplicial 3-complex C with a subcomplex $B \subset C$ so that B is simplicially equivalent to the boundary of a 3-simplex, together with an embedding b of B into $\mathbb{R}^3$. An *embedding* f of (C, B, b) into $\mathbb{R}^3$ is an embedding of C so that $f|_B = b$ and the image of f is exactly the tetrahedron bounded by the image of b. An *isotopy* of a 3-complex with tetrahedral boundary is a continuous family of embeddings.

The goal of this section is to show:

Theorem 14 *There exist a complex with tetrahedral boundary* (C, B, b)*, an embedding* f *of* (C, B, b) *into* $\mathbb{R}^3$*, and a mapping* j *of* (C, B, b) *into* $\mathbb{R}^3$*, such that:*

1. $j|_B = f|_B$,
2. *each vertex in* $C \setminus B$ *is in the strict convex hull of its neighbors,*
3. j *is not an embedding.*

This theorem is a counterexample to the analogue of Tutte's theorem in three dimensions: the existence of f is the analogue of planarity in 2D, the first condition fixes the images of the exterior vertices by j and the second one is the condition for the interior vertices.

The cornerstone for the proof of Theorem 14 is the description by Starbird [30] of a graph C_1, embedded into $\mathbb{R}^3$ in two different ways f_1 and g_1, so that it is impossible to deform one embedding to the other without bending the edges. Yet, if bending the edges is allowed, such a deformation becomes possible. These embeddings are depicted in Figure 11, copied from his paper. We found coordinates for the vertices of these embeddings, available at [1]. In the lemma below, we rephrase the properties stated by Starbird.

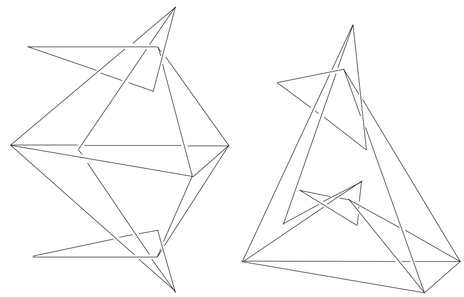

Figure 11: Starbird's embeddings f_1 and g_1 of C_1.

Lemma 15 *1. There are a 3-complex with tetrahedral boundary (C, B, b), so that C contains C_1, and two embeddings f and g of (C, B, b) extending respectively f_1 and g_1.*

2. If C, f and g satisfy the preceding condition, there is no isotopy of (C, B, b) taking f to g.

The first part of Lemma 15 expresses the fact that f and g are combinatorially equivalent triangulations (tetrahedralizations for purists) of a tetrahedron, with the same boundary. Despite this, as stated in the second part, there is no isotopy from f to g. It is to be noted that the analogue of this lemma is false in 2D by Tutte's theorem.

The proof of the second part of this lemma is given in detail in Starbird's paper, we shall not explain the argument here. Shortly said, the author uses properties of piecewise linear curves embedded in 3D space to show that the embeddings f_1 and g_1 cannot be deformed from one to the other while keeping the edges of C_1 straight, for otherwise at some stage of the isotopy there would be a degeneracy which would prevent to have an embedding. Then, because f (resp. g) extends f_1 (resp. g_1), there cannot be any isotopy between those embeddings as well.

We will give a detailed summary of the proof of the first part of Lemma 15, because it is stated in Starbird's paper but not all details of the proof are supplied. The key ingredient for the proof is the following "fundamental extension lemma" enabling to extend an isotopy of a complex to an isotopy of a complex with tetrahedral boundary containing this complex. It is proved in [4, Theorem 3.3]; we rephrase it here for convenience in our framework (it holds in fact in arbitrary dimension):

Lemma 16 *Let C be a simplicial 3-complex and $(h(t))$ be an isotopy of C into $\mathbb{R}^3$. Then there are a 3-complex with tetrahedral boundary $(\tilde{C}, \tilde{B}, \tilde{b})$ so that $\tilde{C}$ contains C and an isotopy $(\tilde{h}(t))$ of $(\tilde{C}, \tilde{B}, \tilde{b})$ into $\mathbb{R}^3$ extending $(\tilde{h}(t))$.*

We shall not give the proof here. The two key ingredients are that slightly perturbing an embedding still yields an embedding, and the use of refinements of triangulations in $\mathbb{R}^3$.

Proof of Lemma 15, first part. We first express the fact that it is possible to deform $f(C_1)$ to $g(C_1)$ if bending the edges is allowed: there is a refinement C_2 of C_1 (by adding vertices on the edges of C_1) and an isotopy $(h(t))$ of C_2 into $\mathbb{R}^3$ taking f_2 to g_2. Here, f_2 is to be understood in the following manner (and similarly for g_2): if v is a vertex in C_1, then $f_2(v) = f_1(v)$; and if an edge $e = vw$ of C_1 is subdivided with vertices $v_0, \ldots, v_n$ inserted on e, then $f_2(v_0), \ldots, f_2(v_n)$ are spread uniformly on $f_1(v)f_1(w)$. It is easy to see that this fact is true, as written in the paper, if you build a model of $f_2(C_2)$ with strings (or small bars) and deform it to $g_2(C_2)$.

No argument apart from the fact that such a deformation is possible is given in Starbird's paper to complete the proof. We thus suggest the following: In fact, we extend a bit more C_2 by protecting each edge of C_1 (split in C_2) by a 3-complex looking like a skinny tube (Figure 12). Define f_2 and g_2 naturally on these tubular protections; the images of f_2 and g_2 are just thickened versions of the images of f_1 and g_1. By Lemma 16, extend C_2 to a 3-complex with tetrahedral boundary (C_3, B_3, b_3), extending the isotopy $(h(t))$ to an isotopy $(\tilde{h}(t))$ of (C_3, B_3, b_3). Now, considering $\tilde{h}(0)$ and $\tilde{h}(1)$, the complex (C_3, B_3, b_3) nearly satisfies the conditions required in the first part of Lemma 15, except that C_3 does not contain exactly C_1 because the edges of C_1 have been subdivided.

Thus, in f_3 and g_3, the only thing we have to do is to retriangulate compatibly the tubular protections of each (split) edge vw of C_1, removing the vertices $v_0, \ldots, v_n$ splitting this edge and restoring the initial edge vw. Since the tubular protections of vw look alike under f_3 and g_3 (the v_i's are on a line, and similarly for the a_i's, b_i's and c_i's), this retriangulation is easy: the compatibility will be

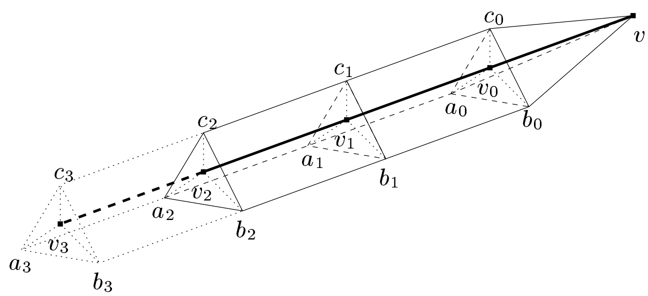

Figure 12: How an edge vw of C_1 (in bold) is protected by a skinny flexible tube. The vertices $v_0, \ldots, v_n$ are spread uniformly on the edge of C_1 which is considered, to make the edge flexible during the isotopy. An equilateral triangle $a_i b_i c_i$ is drawn around v_i, and the vertices of these triangles are linked as shown in the figure. Note the special treatment at the end of the edge (vertex v). The space between the triangles $a_i b_i c_i$ is also triangulated (not all edges are shown in the figure). Thus, a 3-dimensional simplicial complex protects each edge of C_1.

automatically satisfied. See [3, p. 4–6] for similar retriangulation problems: first retriangulate the 2D region which is the convex hull of v, w, and the a_i's by removing the v_i's and linking each of the a_i's to v. Do the same with the b_i's and the c_i's. Now, we have to retriangulate three thirds of the tubular protection of edge vw. To retriangulate the region which is the convex hull of v, w, the a_i's and the b_i's, simply insert a new vertex p in the interior of this region; since its boundary is still triangulated, it is sufficient to insert in the complex the simplexes which are on the boundary of this region with p adjoined ("coning" the boundary of this region from p). Do the same for the other thirds. The resulting complex (C, B, b) and embeddings f and g satisfy the hypotheses. $\square$

Proof of Theorem 14. First notice that, under f and g, all interior vertices are in the strict convex hull of their adjacent vertices. For otherwise a vertex i would be on a face of the polytope generated by the neighbors of i, hence i would have no neighbor on a half-space whose boundary passes through the image of i; this contradicts the fact that i is a vertex interior to the triangulation. Now, because being in the strict convex hull of a set A of points is the same as being a barycenter with positive coefficients of A (Lemma 17), express each interior vertex i as barycenter with positive coefficients of its neighbors, in the embedding f (leading to coefficients λ_{ij}^f for j neighbor of i), and similarly for g. Note that the coefficients may be non-symmetric: we follow the approach of [20] to ensure we have positive coefficients. Then, for $t \in [0, 1]$, consider $\lambda_{ij}^t = (1 - t)\lambda_{ij}^f + t\lambda_{ij}^g > 0$. Fix the positions p_i of the vertices $i \in B$, and look for the positions of the other vertices i satisfying the equations: $\sum_{j|ij \in E} \lambda_{ij}^t (p_j - p_i) = 0$, where E is the set of edges of C. This system admits a unique solution for each $t \in [0, 1]$ (exactly the same proof holds as in Appendix B). Let us call the resulting family of mappings $(\bar{h}(t))$. By Lemma 15, second part, $(\bar{h}(t))$ cannot be an isotopy: there is a $t_0 \in [0, 1]$ such that $\bar{h}(t_0) = j$ is not an embedding. $(C, B, b), f$ and j satisfy the conditions of Theorem 14. $\square$

This theorem is a counterexample to the generalization of Tutte's theorem in 3D, described in introduction. In fact, the result is slightly stronger: j is not an embedding, but even the restriction of j to the 1-skeleton of C is not an embedding (two edges must cross). This also implies that constructing isotopies of complexes in 3D is much more difficult than in 2D. Starbird [31, 32] showed the following theorem which might be a clue to find a solution: if there are two embeddings f and g of a complex K with tetrahedral boundary into $\mathbb{R}^3$ (or more generally if the boundary is a convex polyhedron), then there might be no isotopy from f to g, but there is always a suitable refinement K' of the complex K for which there is an isotopy between f and g. The problem is now to realize algorithmically the refinement and the isotopy; unfortunately, it is unclear how to proceed. Another track would be to try to find more restrictive conditions under which a barycentric method would work; for example, if some subcomplexes are forbidden, or if the complex is sufficiently refined, does Tutte's barycentric method always yield an embedding?

5 Acknowledgements

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A The strict convex hull

Recall that the strict convex hull of a set of points is the interior, in the space affinely generated by this set of points, of the convex hull of these points. The following lemma shows that conditions (ii) and (ii') are equivalent.

Lemma 17 *Let $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$. Then the strict convex hull of A is the set of barycenters with positive coefficients of the points in A.*

Proof. Suppose $p \in \text{Str Conv } A$. Take $k \in \{1, \dots, n\}$. There is an $\varepsilon_k > 0$ so that $p + \varepsilon_k(p - a_k) \in \text{Conv } A$. Therefore it is possible to write $p = \sum_{i=1}^n \mu_i^k a_i$, where $\sum_{i=1}^n \mu_i^k = 1$, $\mu_i^k \geq 0$ and $\mu_k^k > 0$. Taking $\lambda_i = \frac{1}{n} \sum_{k=1}^n \mu_i^k$ yields that p is a barycenter with positive coefficients of the points in A.

For the opposite inclusion, suppose $p = \sum_{i=1}^n \lambda_i a_i$, with $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$. If $|\mu_1|, \dots, |\mu_n|$ are sufficiently small, $p + \sum_{i=1}^n \mu_i(a_i - p)$ is in Conv A. This shows that $p \in \text{Str Conv } A$. $\square$

B Invertibility of System (S)

Lemma 18 *If the coefficients λ_{ij} are positive, System (S) admits a unique solution.*

Before showing this lemma, we must explicitely compute the entries of the matrix involved in System (S). For convenience, note $v_1, \dots, v_m$ the interior vertices and $v_{m+1}, \dots, v_n$ the exterior ones. The matrix involved in System (S) is square, of size m, and defined, if $1 \leq i, j \leq m$ and with the convention $\lambda_{ij} = 0$ if ij is not an edge, by:

$$m_{ij} = -\lambda_{ij}, \text{ if } i \neq j;$$

$$m_{ii} = \sum_{k=1}^n \lambda_{ik}.$$

Several proofs of this lemma exist in the literature. We first give the most straightforward proof in the general case. It uses the well-known "diagonal dominant property" of matrices and can be found in [19, p. 237].

Proof. We show that the kernel of M is $\{0\}$. If $M \cdot y = 0$ for a column vector y with m entries, then: for each $i \in \{1, \dots, m\}, \sum_{j=1}^n \lambda_{ij}(y_i - y_j) = 0$, where $y_j = 0$ if $j > m$ by definition. Consider an index i such that $|y_i|$ is maximal. As λ is positive, the preceding equation yields $y_j = y_i$ for every j neighbor of i. Because G is connected, and because $y_j = 0$ if $j > m$, we get $y_i = 0$. Therefore, M is invertible. (In fact, the same argument shows that M is symmetric definite positive, for it cannot have a nonpositive eigenvalue). $\square$

We now prove Lemma 18 in the special case where the coefficients are symmetric, using the physical interpretation with the springs. E_i denotes the set of interior edges.

Proof. The energy of the system made of the springs is defined by $\mathcal{E} = \frac{1}{2} \sum_{ij \in E_i} \lambda_{ij} |p_j - p_i|^2$. Consider that the positions of the exterior vertices are fixed; $\mathcal{E}(p_1, \dots, p_m)$ is a polynomial function of degree two. If at least one interior vertex p_i goes to infinity, $\mathcal{E}$ tends to $+\infty$ by connectivity of G and positivity of the coefficients. Thus, the homogeneous polynomial of degree two in the coordinates $p_1, \dots, p_m$ of $\mathcal{E}$ is a quadratic form which is symmetric definite positive. But the matrix of this quadratic form is exactly the matrix M, as it can be checked easily using the fact that the coefficients are symmetric. Thus M is symmetric definite positive and (S) admits a unique solution. $\square$

Finally, we indicate that Lemma 18 is a consequence of the *matrix tree theorem* (see Brualdi and Ryser [7, p. 324], Chaiken [9], Orlin [28] or Zeilberger [41]), a theorem interpreting combinatorially the determinant of certain matrices in terms of arborescences of graphs.

Proof. Let $(n_{ij})_{1 \leq i \neq j \leq m+1}$ be real numbers. Consider the complete directed graph (without loops) $\bar{G}$ with $m+1$ vertices, each edge (ij) having, by definition, weight n_{ij}. Let P be the square matrix of size $m+1$ defined by:

$$p_{ij} = -n_{ij}, \text{ if } i \neq j;$$

$$p_{ii} = \sum_{k=1}^{m+1} n_{ik}.$$

The matrix P is called the *Laplacian matrix* of $\bar{G}$. A *spanning arborescence* of $\bar{G}$ rooted at i is a subgraph of $\bar{G}$ covering all vertices of $\bar{G}$ so that it has no directed cycle and all vertices $j \neq i$ have, in $\bar{G}$, outdegree equal to one. The *matrix tree theorem* asserts that the cofactor of the ith diagonal element of matrix P is exactly the sum, over all spanning arborescences of $\bar{G}$ rooted at i, of the product of the weights of the edges of this arborescence.

Apply this theorem to our particular case: let $n_{ij} = \lambda_{ij}$ if $1 \leq i \neq j \leq m$; if $i \leq m$, let $n_{i,m+1} = \sum_{k=m+1}^{n} \lambda_{ik}$ and $n_{m+1,i} = 0$. The $(m+1)$th cofactor of P is exactly the determinant of the matrix M and also equals the sum, over all spanning arborescences of $\bar{G}$ rooted at vertex $m+1$, of the product of the weights of the edges of this arborescence. There is at least one spanning arborescence yielding a nonzero contribution to this sum: to see this, take a spanning tree of the graph induced by the inner vertices of G, and add one directed edge from a vertex in G which, in G, is linked to an exterior vertex, to vertex $m+1$. Since the weights of the edges are nonnegative, the contribution of any spanning arborescence is nonnegative, hence the cofactor is positive and M is invertible. $\square$