Tutte's Barycenter Method applied to Isotopies

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Abstract

This paper provides a short and intuitive proof of Tutte's barycentric embedding theorem [14], compared to the original one which involved a lot of graph theory and complicated terminology. We then study a method closely related to papers by Gotsman, Surazhsky and Floater [8, 9, 10] to build isotopies of triangulations in the plane, whose theoretical basis is Tutte's theorem. Our method works on a smaller class of embeddings, but has a clear physical interpretation. We finally show that Tutte's result does not extend to 3D space, reviewing also a paper by Starbird [12].

1 Introduction

Tutte's theorem. In 1963, Tutte [14] gave a way to build embeddings of any planar, 3-connected graph $G = (V, E)$. Let C be a cycle whose vertices are the vertices of a face of G in some (not necessarily straight-line) embedding Γ' of G. Let Γ be a straight-line mapping of G into the plane, satisfying the conditions:

i. the set V_e of the vertices of the cycle C is mapped to the vertices of a strictly convex polygon Q, respecting the order of the points;

ii. each vertex in $V_i = V \setminus V_e$ has to be a barycenter with positive coefficients of its adjacent vertices (Tutte assumed all coefficients to be equal to 1, but the proof extends without changes to this case). In other words, choose $\lambda_{uv} > 0$ for each vertex $u \in V_i$ and each neighbor v of u. If $\overline{v}$ denotes the image of vertex v under Γ, the system (S): for each $u \in V_i, \sum_{v|uv \in E} \lambda_{uv}(\overline{u} - \overline{v}) = 0$ has to be satisfied (it can be shown easily that it has a unique solution, see [7, p. 237]).

Theorem 1 (Tutte's Theorem) *Γ is an embedding of G into the plane, with strictly convex inner faces.*

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For a survey on graph drawing methods, see [6]. We give a new and simpler proof of Tutte's theorem, first showing his result under two additional restrictions:

iii. the graph G is triangulated: every face of Γ', except possibly the face bounded by C, is a triangle;

iv. the images of these triangles under the mapping Γ are non-degenerate: their interior is non-empty.

Proving non-degeneracy (under hypotheses (i) and (ii), three vertices belonging to a face are not on the same line) seems to be the core of the problem and is done quite easily with this first step, using the non-planarity of the bipartite graph $K_{3,3}$ and simple geometric ideas. Our proof is primitively inspired by Tutte's paper, but differs a lot by its structure and involves far less background in graph theory. Other proofs of this theorem exist in the literature. Becker and Hotz [2] use the notion of "quasi-planarity" as limit case of a planar situation, yielding to unnecessary complications. Y. Colin de Verdière [5] shows the result on arbitrary surfaces of non-positive curvature using the Gauss-Bonnet formula, but his proof is valid only for triangulated graphs and does not seem to extend to 3-connected graphs.

Isotopies. Tutte's theorem yields a method, described by Gotsman et al. in [8, 9, 10], to morph two triangulations, the boundary being the same convex polygon in both embeddings. One can compute coefficients $\lambda_{uv} > 0$ for each interior vertex u and each neighbor v of u so that u is the barycenter with coefficients $(\lambda_{uv})_v$ of its neighbors in the initial embedding. Doing the same for the final embedding and interpolating linearly the coefficients yields an isotopy (a continuous family of embeddings) by Tutte's theorem. We study the symmetric case ($\lambda_{uv} = \lambda_{vu}$). The interest of this approach is that it has a nice physical interpretation: fix the exterior vertices and edges and replace each interior edge joining two vertices u and v by a spring with rigidity λ_{uv}; then the equilibrium state of this physical system is the solution of system (S). Spring models of graphs have been used in several other contexts. We rely on the Maxwell-Cremona theorem from rigidity theory to compute the coefficients. The drawback of our method is that they are not always positive, hence Tutte's theorem does not apply in all cases. After small experiments (with 20 vertices or so), we thought that, despite this, our method always yielded an isotopy. It is not the case, and we have small examples refuting this conjecture. However, we can compute isotopies if both embeddings are regular triangulations.

Generalization to 3D space. Finally, we study the extension of Tutte's theorem to three dimensions. We shortly present the proof of the existence of two triangulations of a tetrahedron which are combinatorially equivalent but yet for which there is no linear isotopy from one to the other, a fact which is specific to spaces of dimension ≥ 3 shown by Starbird in [12]: we give an outline of the proof and explain parts of the proof not detailed in his paper. Then we prove that the natural generalization of Tutte's theorem in 3D (where we consider mappings and embeddings of a 3-simplicial complex into a convex polyhedron) is false. To our knowledge, this attempt of generalizing Tutte's theorem is new, and our refutation of this extension raises interesting open questions, in the context of isotopies as well as in view of embedding 3-complexes.

2 Proof of Tutte's theorem

We prove here Tutte's theorem [14], using the notations of the introduction. Define the *strict convex hull* of a set of points to be the interior, in the space affinely generated by these points, of the convex hull of these points; (ii) is equivalent to "each $v \in V_i$ lies in the strict convex hull of its neighbors". A *face of G* (or *triangle* if (iii) is satisfied) is the set of vertices and edges of a face of the embedding Γ' except the face bounded by C. Note that, in the situation of the theorem, each $v \in V_i$ is mapped by Γ into the interior of Q.

Lemma 2 *Assuming (iii) and (iv), Theorem 1 holds.*

Proof. We here only show the fact that two triangles sharing a common vertex cannot overlap under Γ. By (iv), the angles of the triangles of Γ are well-defined and positive. Let $\alpha(v)$ be equal to 2π if $v \in V_i$, and to the angle of the polygon Q at v if $v \in V_e$. Let $\sigma(v)$ be the sum, over the triangles incident to v, of the angle of the image of such a triangle at v in the mapping Γ. By (ii), it is easy to see that $\sigma(v) \geq \alpha(v)$, with equality if and only if there is no "folding" at v. But one can show that $\sum_{v\in V} \alpha(v) = \sum_{v\in V} \sigma(v)$ by applying Euler's formula to Γ' and using the fact that the sum of the angles of a triangle equals π, hence $\sigma(v) = \alpha(v)$ for each v. □

Now, under the hypotheses of Theorem 1, our goal is to show that some degenerate cases cannot occur. In the following general lemma, paths are supposed to be simple and non-degenerate.

Lemma 3 *Let v_1, v_2, v_3 and v be pairwise distinct vertices of a graph H. Assume, for $i = 1, 2, 3$, that there is a path P_i from v_i to v which avoids the v_j's (for $j \neq i$). Then there exist three paths P'_i, from v_i to a common vertex v', which are pairwise disjoint (except at v').*

From this point, φ is a non-constant affine form of the plane, representing a line by its zero set. A vertex v of G is *φ-active* if it has a neighbor v' with $\varphi(\overline{v}) \neq \varphi(\overline{v}')$, *$\varphi$-inactive* otherwise. The *φ_+-poles* are the vertices $v \in V$ with $\varphi(\overline{v})$ maximal. A φ_+-pole must be in V_e, and there are one or two φ_+-poles. If $\overline{v}_1, \ldots, \overline{v}_k$ lie on the line $\varphi = 0$, $G(\varphi_+, v_1, \ldots, v_k)$ is the graph induced by the vertices lying in the half-plane $\varphi > 0$, to which we add the vertices $v_1, \ldots, v_k$ and all edges from one of these vertices to a vertex in $\varphi > 0$. Let $G(\varphi)$ be the subgraph of G induced by the vertices v lying on the line $\varphi = 0$.

Lemma 4 *Let v be a φ-active vertex so that $\varphi(\overline{v}) = 0$; assume that v is not a φ_+-pole. Then there exists a path in $G(\varphi_+, v)$ from v to a φ_+-pole of G.*

The two following lemmas show that some degenerate cases cannot occur. The first one, whose proof is omitted, is along the lines of [14], contrary to the second one which uses another crucial argument.

Lemma 5 *For any φ, G has no φ-inactive vertex.*

Lemma 6 *Let v_i, $i = 1, 2, 3$, be three vertices of a face of G. Then, under Γ, the v_i's are not collinear.*

Proof. Assume the v_i's are on the line $\varphi = 0$. The v_i's are φ-active and not φ_+-poles. By Lemmas 4 and 3, there exist two vertices x and y and six disjoint paths Q_i joining each of the v_i's to x and y. Let G' be the graph G with an additional vertex w linked to the v_i's. G' is planar (the v_i's belong to a common face) but contains a subdivision of $K_{3,3}$, which is impossible. □

We now give a sketch of the proof of Theorem 1. By Lemmas 6 and 2, Theorem 1 is true if the faces of G are triangles. The idea is to triangulate the faces of G to get a graph G_1 and a mapping Γ_1, which is triangulated hence an embedding, and to go back to the original mapping. Application of Lemma 5 to Γ is necessary in the process to check condition (ii) for Γ_1.

3 Isotopies in the plane

Now, we detail the construction of the isotopy. Let $G = (V, E)$ be a 3-connected planar graph and Γ_0 and Γ_1 be two embeddings of G into the plane so that: the boundary cycle C of the outer face of Γ_0 is a convex polygon, it bounds also the outer face of Γ_1, and the corresponding vertices of C are at the same location in Γ_0 and Γ_1. We look for an isotopy between Γ_0 and Γ_1, during which the vertices of C remain at the same position. In addition, we will require G to be triangulated.

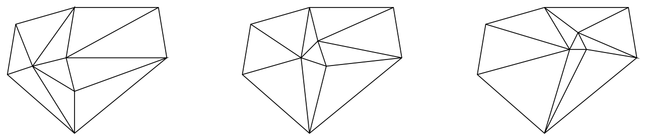

Figure 1: Steps Γ_0, $\Gamma_{1/2}$ and Γ_1 of an isotopy.

Let E_i be the set of (undirected) interior edges. A *weight function* on Γ, or *stress*, is a map $\omega : E_i \to \mathbb{R}$ (thus $\omega_{uv} = \omega_{vu}$). ω is *positive* if $\omega_{uv} > 0$ for each interior edge uv. If ω and the positions of the exterior vertices are fixed, the *equilibrium state* is defined by the system: for each $u \in V_i, \sum_{v|uv\in E} \omega_{uv}(\overline{u} - \overline{v}) = 0$. In these conditions, ω is an *equilibrium stress* for Γ.

Here is a summary of our approach, whose physical interpretation has been given in the introduction: compute equilibrium stresses ω^0 (resp. ω^1) of embeddings Γ_0 (resp. Γ_1); then, for $t \in [0,1]$, compute the equilibrium state of $\omega^t = (1-t)\omega^0 + t\omega^1$. The difficulty resides in computing an equilibrium stress for a given embedding Γ: our method relies on the Maxwell-Cremona correspondence, a theorem from rigidity theory (see [11]). Think of Γ as being in the plane $z = 0$ of $\mathbb{R}^3$. Take any *lift* of Γ, by adding to each vertex $\bar{v} = p_v = (x_v, y_v, 0)$ of Γ a third coordinate, leading to $q_v = (x_v, y_v, z_v)$. If $a_0, \ldots, a_k$ are $k+1$ points of $\mathbb{R}^k$, written in column vectors, we note $[a_0, \ldots, a_k] = \begin{vmatrix} a_0 & a_1 & \ldots & a_k \\ 1 & 1 & \ldots & 1 \end{vmatrix}$.

Theorem 7 *For each $ij \in E_i$, consider its left and right neighbors l and r; let* $\omega_{ij} = \dfrac{[q_i, q_j, q_l, q_r]}{[p_i, p_j, p_l][p_i, p_j, p_r]}$. *Then ω is an equilibrium stress for Γ.*

If Γ_0 and Γ_1 are regular triangulations, then we can compute an isotopy between Γ_0 and Γ_1. Indeed, ω is positive if and only if the lift of Γ is a lower convex hull. If ω^0 and ω^1 are positive, the stresses $\omega^t = (1-t)\omega^0 + t\omega^1$ yield an isotopy by Tutte's theorem.

In practice, we tried to build an isotopy between a random triangulated embedding and the embedding of the same graph obtained by Tutte's method when all weights equal 1. We lift Γ_0 to the standard paraboloid $z = x^2 + y^2$, compute the equilibrium stress, and linearly interpolate ω^0 and $\omega^1 = 1$. Although the initial stress is not necessarily positive, this method yields an isotopy in many (not too big) cases; long experiments have been necessary to find a counter-example like Figure 2. Our smallest counter-example uses 4 outer vertices and 2 inner vertices, but the failure cannot be seen on the screen and can only be proved by computation. This method has been implemented in C++ using Numerical Recipes, and also Mathematica for exact computations. Examples are available at [1].

The study of the stresses which yield an embedding remains to be done. Are there Γ_0 and Γ_1 so that, for any equilibrium stresses ω^0 and ω^1, the interpolation $\omega^t = (1-t)\omega^0 + t\omega^1$ does not yield an isotopy? When using linear interpolation with lifts on the standard paraboloid, we experimentally noticed that the matrix of system (S) is always symmetric definite positive, but have no proof of this fact.

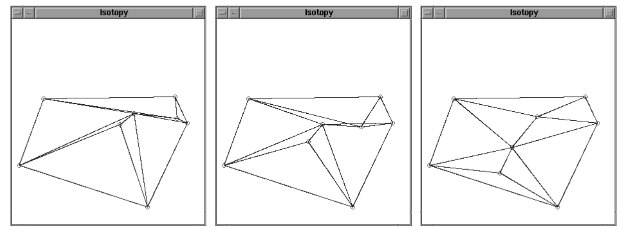

Figure 2: Non-planarity can occur with this method.

4 Generalization to 3D space

We explain here why the analogous of Tutte's theorem is false in 3D space, first reviewing a paper by Starbird [12]. The cornerstone of Starbird's paper is a graph C with two embeddings f and g of C into $\mathbb{R}^3$, so that is possible to deform one embedding to the other only if bending the edges is allowed (see Figure 3, copied from his paper). We found coordinates for the vertices, available at [1]. Here are a few definitions. A *3-complex with tetrahedral boundary* (C', B', b') is a 3-simplicial complex C' with a subcomplex $B' \subset C'$ so that B' is (combinatorially) the boundary of a tetrahedron, together with an embedding b' of B' into $\mathbb{R}^3$. An *embedding* f' of (C', B', b') into $\mathbb{R}^3$ is a (straight-line) embedding of C' so that $f'|_{B'} = b'$ and the image set of f' is the tetrahedron bounded by the image set of b'.

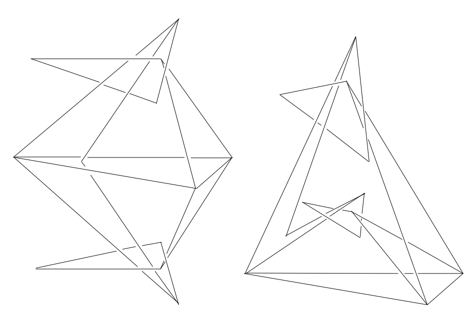

Figure 3: Starbird's embeddings f and g of C.

Lemma 8 (rephrased from [12]) *There are a 3-complex with tetrahedral boundary (C', B', b') so that C' contains C and two embeddings f' and g' of (C', B', b') extending respectively f and g.*

In the situation of Lemma 8, Starbird shows that there is no isotopy of (C', B', b') taking f' to g'. However, Lemma 8 says that f' and g' are combinatorially equivalent triangulations of a tetrahedron, with the same boundary. The analogous of this situation cannot occur in 2D by Tutte's theorem. We shall now prove Lemma 8, detailing facts not given in Starbird's paper.

Proof. First, it is easy to check that it is possible to deform $f(C)$ to $g(C)$ if bending the edges is allowed: there is a refinement C_r of C (by adding vertices on the edges of C) and an isotopy (h_t) of C_r into $\mathbb{R}^3$ taking f_r to g_r, where f_r (and similarly for g_r) is defined by: if v is a vertex in C, then $f_r(v) = f(v)$; if an edge $e = vw$ of C is subdivided with vertices $v_0, \dots, v_n$ inserted on e, then $f_r(v_0), \dots, f_r(v_n)$ are spread uniformly on $f_r(v)f_r(w)$.

Theorem 3.3 of [3], by Bing and Starbird, implies that it is possible to extend C_r and (h_t) to a 3-complex with tetrahedral boundary (C'_r, B', b') and an isotopy (h'_t) of (C'_r, B', b'). Now, considering h'_0 and h'_1, the complex C'_r nearly satisfies the hypotheses of Lemma 8, except that C'_r does not contain exactly C because the edges of C have been subdivided. No precise remedy to this problem is discussed in Starbird's paper; we propose the following.

Define C_r by splitting edges of C, as above, but also by protecting each edge of C by a 3-complex looking like a tube (Figure 4). C_r is a thickened, flexible version of C; there still exists an isotopy between f_r and g_r (with obvious notations). As above, we get a complex with tetrahedral boundary (C'_r, B', b') and embeddings f'_r and g'_r of (C'_r, B', b') into $\mathbb{R}^3$. In f'_r and g'_r, the tubes protecting each (split) edge of C look alike and we can retriangulate them compatibly in f'_r and g'_r, removing the splitting vertices. The resulting complex C' and embeddings f' and g' satisfy the hypotheses. $\square$

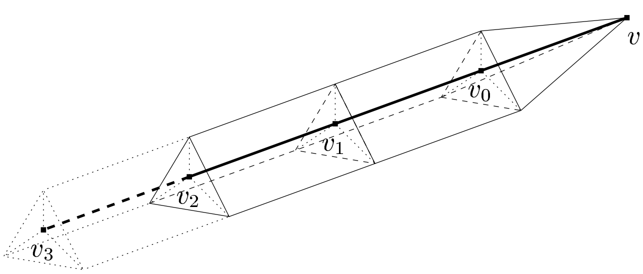

Figure 4: Each edge of C (in bold) is protected by a flexible 3-complex: a triangle T_i is drawn around each splitting vertex v_i and the vertices of these triangles are linked as shown in the figure. Spaces between triangles are also triangulated (not all edges have been drawn).

Theorem 9 *With the notations of Lemma 8, there is a mapping j' of C' into $\mathbb{R}^3$, with $j'|_{B'} = f'|_{B'}$, so that the image of each vertex in $C' \setminus B'$ is, by j', in the strict convex hull of its neighbors and j' is not an embedding.*

Proof. Under f' and g', each interior vertex u is in the strict convex hull of its neighbors, hence is a barycenter with positive coefficients of its neighbors. Interpolating linearly these barycentric coefficients of f' and g' yields a family of continuous mappings; because there is no isotopy from f' to g', one of these mappings j' is not an embedding, hence the proof is complete. $\square$

This theorem is a counter-example to the generalization of Tutte's theorem in 3D, which also implies that constructing isotopies of 3-complexes is much more difficult than in 2D. However, Starbird [13] showed that, if there are two embeddings f and g of a complex with tetrahedral boundary into $\mathbb{R}^3$, then there is a refinement of this complex K for which there is an isotopy between f and g. But the algorithmic realization of the refinement and the isotopy looks difficult. Maybe a barycentric method could work under more restrictive conditions, for example if some subcomplexes are forbidden or if the complex is sufficiently refined.

Acknowledgements

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