VORONOÏ DIAGRAMS IN PROJECTIVE GEOMETRY
AND SWEEP CIRCLE ALGORITHMS FOR CONSTRUCTING
CIRCLE-BASED VORONOÏ DIAGRAMS

- extended abstract -

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The sweep circle algorithm is based on the concept of wavefront developed by Dehne and Klein [Dehne, 1997]. In this case, because the sweep line is a site of the same nature as the sites being considered, we get an economy of scale for the objects to be treated. It uses the definition of a Voronoï diagram in projective geometry to simplify the treatment of the non-connected wavefront and vertex events.

The paper gives an application of this kind of algorithm to the computation of a subtractively weighted Voronoï diagram. However the properties of the sweep circle algorithm in projective geometry can be used to construct other circle-based Voronoï diagrams and another application of these concepts is being developed for the construction of a Voronoï diagram of circles in Euclidean geometry.

All sites are considered in general position.

1. The subtractively weighted Voronoï diagram.

This Voronoï diagram, where a weighted point site is represented by a disk, was studied in great detail by Sharir [Sharir 1985]. Two facts have to be underlined here.

The first one is that when a disk is contained in some other disk the latter site dominates the former and the former site has no Voronoï region of its own. Therefore if we do a shrinking circle sweep starting with a circle of infinite radius, *there will be a natural masking of all the sites inside the sweeping circle*, which will discover the sites as the radius decreases. Moreover since this sweeping circle is a site by itself, if we want to construct an ***n***-sites Voronoï diagram, we will in fact have, at any time during the algorithm, an **(*n*+1)**-sites Voronoï diagram, which will be fully constructed in the whole plane. Finally, if the location of the sweep centre is well chosen (inside one and only one site) the sweep circle and the wavefront will disappear altogether at the end of the sweep because the site which contains the centre of the sweep will dominate the sweep circle when its radius is small enough.

The second one is that a subtractively weighted Voronoï diagram can be non-connected. Therefore *the wavefront*, which is also part of a temporary subtractively weighted Voronoï diagram, *may also be non-connected at times*. In order to have an optimal algorithm, it is necessary to manage wisely the wavefront when a non-connected edge is created or merged with another one.

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2. Shrinking circle sweep.

This algorithm is a generalization of the Fortune algorithm [Fortune 1987]; the quick description which follows gives only the main differences with it (the figures speak for themselves). The justifications are given in the following paragraph on a projective interpretation of a Voronoï diagram.

The centre being chosen as indicated, the sweep starts with a circle dominating all others. The first type of event is of course the discovery of a site, the difference with a line sweep occurring if the Voronoï diagrams and, therefore, the wavefront is non-connected (Fig. 1- a).

The second type of event is merging together two wavefronts. It happens when the sweep circle is tangent to the common tangent of two sites (Fig. 1- b). Finally the third type of event is a vertex event, the greatest empty circle being inside three sites (Fig. 1- c) or outside all of them since its centre, equidistant from all of them, is at a weighted-distance from each which has obviously the same sign. At this moment of the sweep six bisectors (all hyperbolas) are concurrent: three are part of the temporary Voronoï diagram, three belongs to the final diagram.

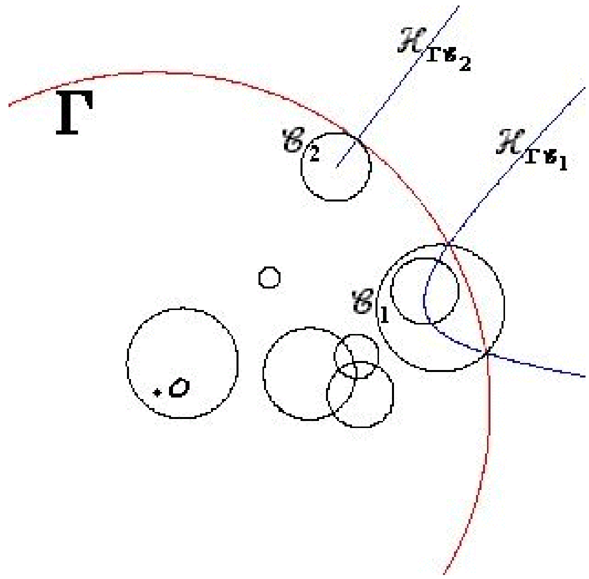

Fig. 1-a

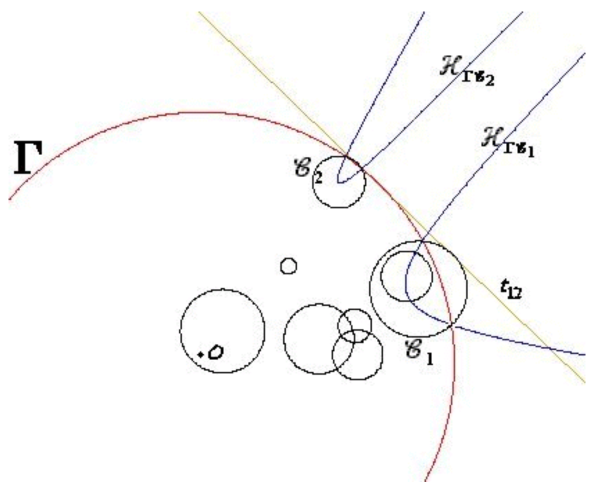

Fig. 1-b

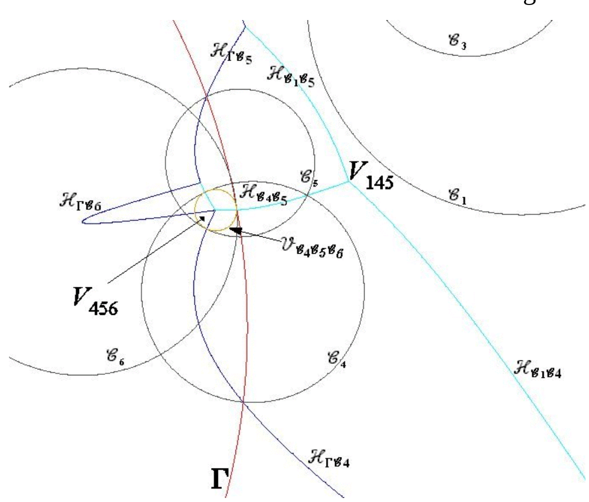

Fig. 1-c

3. *Voronoï diagrams in projective geometry.*

The definition of a Voronoï diagram in projective geometry may not seem useful at first sight because there is no distance involved. However it can be done easily and makes it possible to solve efficiently the problem of the infinite elements of the diagrams. The generalization of the properties of the Voronoï diagram is based on the fact that the points belonging to edges or vertices of Voronoï diagrams are centres of empty circles tangent to two or three sites.

In the projective plane the sites are circles. A point is on an edge of a Voronoï diagram if it is the centre of a circle tangent to two sites and not containing any other point from other sites. The centre of a circle is

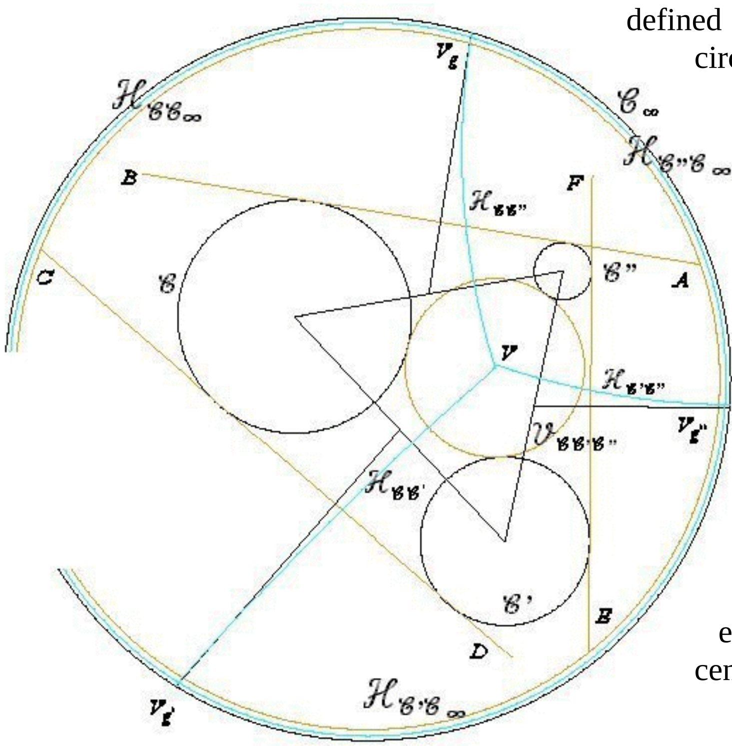

Fig. 2

defined as the envelope of the normal to the circle, and a circle is defined as a second-order algebraic curve going through the cyclic points of the plane. This means that there are four types of circle: the normal one, the point (two isotropic lines), any line and the line at infinity of the plane and the double line at infinity (the equation $Z^2 = 0$). The centre of a circle is the pole of the line at infinity in the first two cases, the harmonic conjugate of the intersection of the two lines relative to the cyclic points in the second case and all the points of the line at infinity in the last one. This explains the unusual definition of the centre given above.

The definition of a point inside a conic is the usual one in projective geometry (by checking the order of the intersection points with the conic by two lines passing through the point to be tested). Since any projective transformation will conserve the harmonic ratio, an empty circle in Euclidean geometry will conserve this property in the projective plane. The interior of the degenerate circle formed by one straight line and the line at infinity of the plane has no meaning, but an empty half-plane has, and if the straight line creates an empty half-plane containing the centre of such a circle (which is on the line at infinity) it determines an empty circle and if it is the greatest empty circle, the centre is therefore considered as a Voronoï vertex at infinity.

With these definitions one can embed a Voronoï diagram of circles from a Euclidean plane to a projective one (Fig. 2) *as long as a special site, a degenerate circle formed by the double line at infinity* $\mathbf{C}_h$**,** *is added to the sites*. This new site allows for an easy projective interpretation of the whole diagram. The element at finite distance are unchanged (sites **C**, **C'** and **C"**, greatest empty circle $\mathbf{V}_{\mathbf{CC'C''}}$, vertex V, edges $\mathbf{H}_{\mathbf{CC'}}$, $\mathbf{H}_{\mathbf{C'C''}}$, $\mathbf{H}_{\mathbf{CC''}}$) but one sees immediately that any part of the convex envelope that is a straight line (e.g. AB) is part of the greatest empty circle tangent to the two usual sites and the site $\mathbf{C}_h$. The

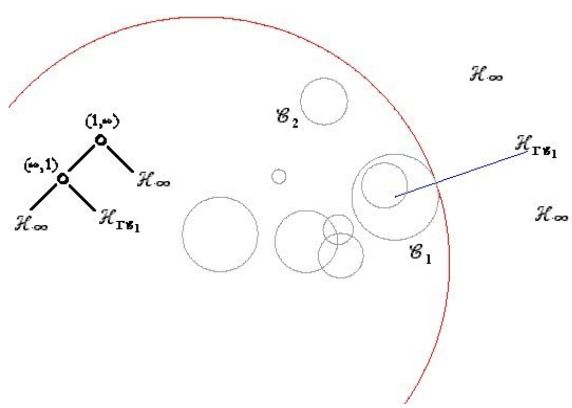

Fig.3-a The figures show a non-connected wavefront and a vertex event at infinity. The binary tree is not balanced to enable the reader to follow closely the figures

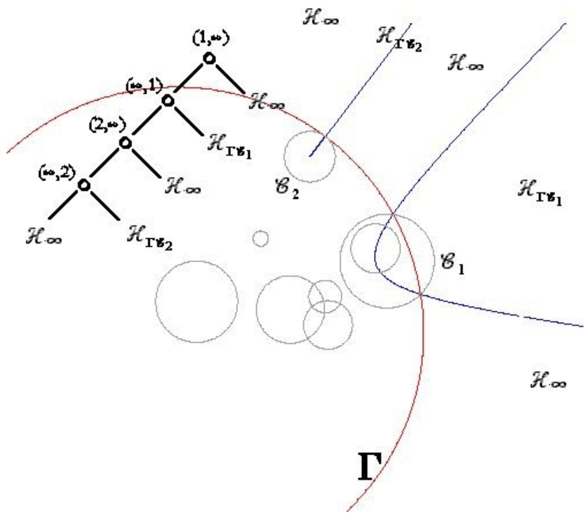

Fig 3-b

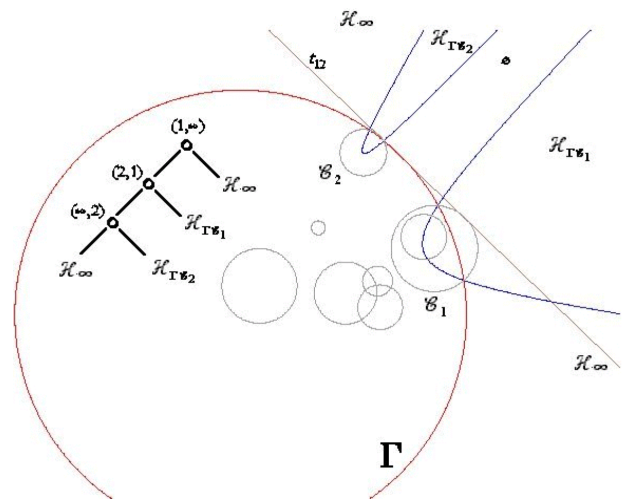

Fig. 3-c

vertices at infinity V_g , $V_{g'}$, $V_{g''}$ are the points in the direction perpendicular to the common tangents; they are also the points at infinity of the hyperbolic edge of the Voronoï diagrams separating the two usual sites. The edges at infinity $\mathbf{H}_{CC_h}$, $\mathbf{H}_{C''C_h}$, $\mathbf{H}_{C''C_h}$ separate the usual site and the special site $\mathbf{C}_h$ *which therefore does not modify the Voronoï diagram at finite distance.*

The main purpose of this embedding is to facilitate the treatment of the event of the second type when two non-connected wavefronts are merged or of the first type when the discovery of a site leads to a non-connected wavefront.

The Voronoï diagram which is not necessarily connected in the Euclidean plane is connected through the edges at infinity in the projective plane because of the addition of the special site. The events of the second type are nothing else but a “vertex at infinity event”, and therefore, with the introduction of projective geometry, it is not a special case anymore. The binary tree used to follow the wavefront and the priority list to manage the events become totally similar to what is done for the Fortune algorithm (Fig. 3-a,b,c). With the exception of the treatment of the false alarms (see below) the overall structure of both algorithms is identical. The complexity of the algorithm is therefore optimal.

4. False alarms - sterile events.

The computation of vertex events in polar coordinates is done using Casey’s condition (a generalization of Ptolemy’s theorem [Coolidge, 1917]). This theorem gives the condition for four circles to be tangent to a fifth. The equation is a condition on common tangential segment ($t_{CC'}$) between two circles C, C' (Fig. 4). It is interesting to use because, when one gets three neighbouring sites C, C', C'' and the mobile sweep site Γ, there is no need to know anything about the fifth one (which is the greatest empty circle) in order to compute the position of the corresponding vertex event. Casey's condition is expressed by:

$$t_{\Gamma C}\cdot t_{C'C''} \pm t_{\Gamma C'}\cdot t_{CC''} \pm t_{\Gamma C''}\cdot t_{CC'} = 0$$

On the negative side, this equation of the fourth order in the radius of the sweeping circle gives more solutions than necessary (geometrically there are eight at the most, which are given by the roots of the equation and the choice of $\pm$).

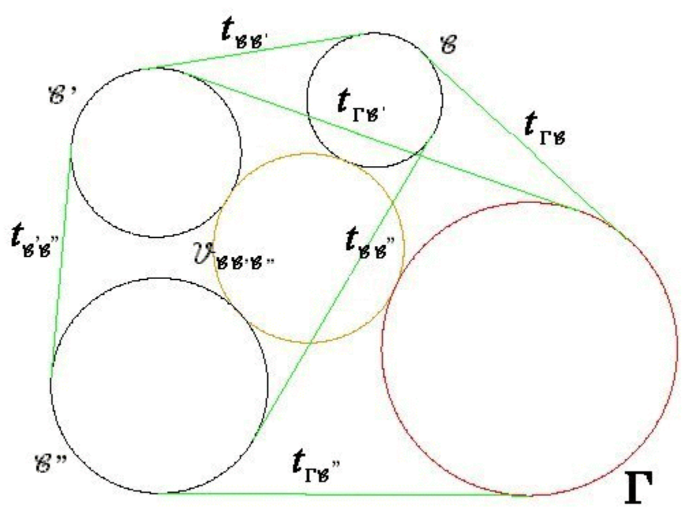

Fig. 4 Γ is the sweep circle, the three sites are **C**, **C'** and **C"** and **$V_{CC'C''}$** is the greatest empty circle

Since the number of these false alarms is bounded, no attempt is made to eliminate these solutions. Therefore this does not increase the complexity of the algorithm, only the multiplicative coefficient. The algorithm eliminates the false alarms by considering the length of the edge of the front wave which has to disappear at the vertex event before dealing with it. If the length is nil, then the edge will disappear and the vertex event is processed, if not the event is discarded before any other processing. This is easily checked by considering the fact that the coordinates of the cusps of the front wave are in fact parametric representations of the edges of the diagram: the event is an actual vertex event if and only if the numeric values of both parametric equations, given the radius of the sweep circle at that time of the sweep, is the same. When this happens, the two cusps are merged in a single vertex; if not, the event is sterile, which means that even if it were processed, it would not yield any new information.

This choice being made for the parasite solutions of Casey's condition, all false-alarms which do not eliminate themselves before reaching the top of the event queue, are processed the same way: in any case the number of events which can be in the event queue at any time depends linearly upon the number of edges of the front wave; so leaving them in the queue will not change the complexity of the algorithm.

The gain is rather a gain in simplicity, because the elimination of false alarms given by this method would be time-consuming. This is the main difference with the way the Fortune's algorithm is carried out by Guibas and Stolfi [Guibas, 1988] which take care of eliminating upfront all false-alarms through a "death pointer".

5. Elimination of dominated sites.

The elimination of a site dominated by another one does not offer any difficulty. When a new site is discovered a test is done on the localization of its centre. If the centre is outside the wavefront, the site is eliminated. The circle $\mathbf{C}_j$, centre C_j, is included in the circle $\mathbf{C}_i$, centre C_i, if and only if, ρ, being the radius of the sweep circle Γ, satisfies the inequality: $OC_j - C_iC_j > \rho - r_i$ (Fig. 5).

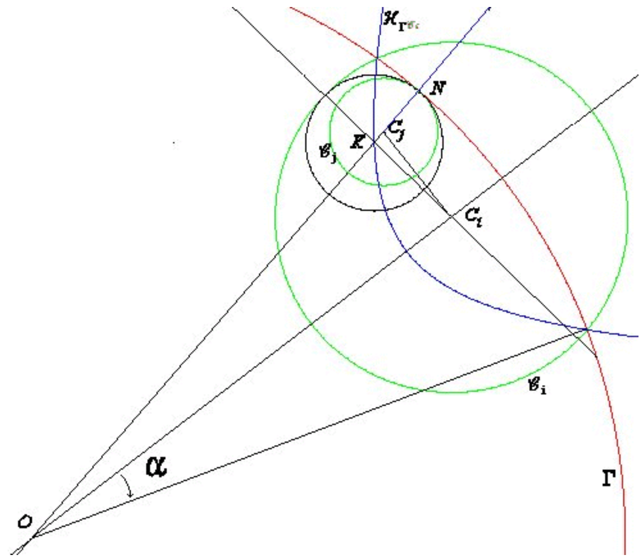

Fig. 5

6. No-vertex Voronoï diagrams.

The usual Fortune's line sweep algorithm can be easily interpreted in the projective plane: it is a circle sweep, with a circle of the third type (a line and the line at infinity) whose centre is not within a site but on it. A difficulty arises from the facts that the variable site and the special site have a common part ($Z = 0$) and that the sweep circle does not disappear at the end of the sweep: the vertices at infinity, which are on the circle from the start to the end, cannot be discovered. This explains why the parabolic front does not disappear either at the end of the sweep and that a additional, though simple, post-processing algorithm is needed for these vertices.

In the case of subtractively weighted Voronoï diagrams, it has been shown by Sharir [Sharir, 1985] that it is possible to have a complete diagram with no vertices (Fig.6 - ellipses excluded)). Fortune's algorithm is not very interesting here because it is executed for no other purpose than to show that the Voronoï diagram to be constructed has no vertices. The construction of the diagram has to be done at the end of the algorithm. As seen above this no-vertex diagram is a diagram with vertex at infinity only. With the sweep circle algorithm in projective geometry these vertices are treated during the sweep as usual (as events of the second type); therefore there is no need for additional work at the end of the algorithm. The natural masking used for the sweep by the shrinking circle makes any additional work at the end of the sweep unnecessary.

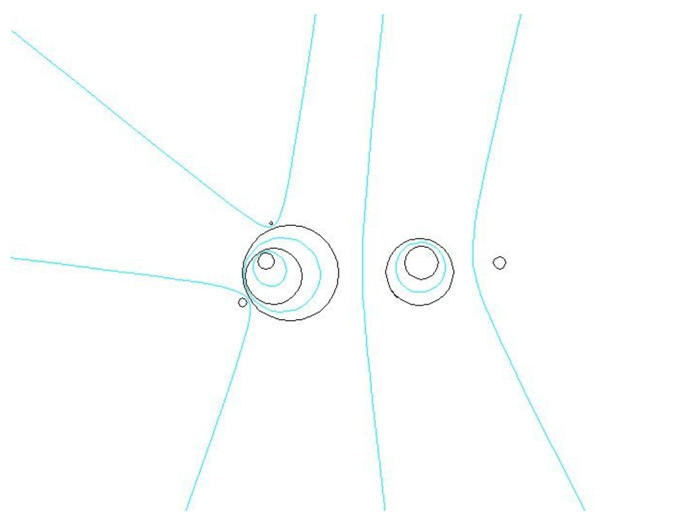

Fig. 6

7. Conclusions.

The use of projective geometry unifies the treatment and simplifies the exceptional case where a vertex is at infinity. It gives of course the convex envelope of the circles at the same time. One can notice that sweep circle algorithms are more often considered useful to process the Voronoï diagram locally [Dehne, 1988], [Adam, 1997] however with this point of view it is possible to handle all sites. The algorithm can be used to deal only with points, but then it has already been done [Adam, 1997], [Adam, 1998]. The same concept is used to develop an optimal algorithm to construct Voronoï diagrams for circles in euclidean geometry, be they no-vertex diagrams (Fig. 6) or with a complexity that can reach $O(n^2)$.

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