Guarding the Convex Subsets of a Point-set
(Extended Abstract)

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Abstract

A point in the plane is called a *guard* for the convex set C if it lies in the interior of C. Let $\mathcal{P}$ be a planar point-set. A set $\mathcal{S}$ of points is a *k-convex guard set for $\mathcal{P}$* if every convex k-gon formed from points of $\mathcal{P}$ contains a guard from $\mathcal{S}$. We study, for any integer $k \geq 3$, the minimum size of a k-convex guard set of a given planar point-set of size n. We give the tight bounds for the case $k = 3$, for any arrangement of n points, c of which form their convex hull. We prove that, in this case, $2n - c - 2$ guards are always necessary and sufficient to cover all the triangles having vertices from $\mathcal{P}$.

1 Introduction

In this paper we consider planar point-sets with points in general position (i.e., no three points in the set lie on a straight line). A point in the plane is called a *guard* for the convex set C if it lies in the interior of C.

Definition 1.1 *Let $\mathcal{P}$ be a planar point-set on n points. A set $\mathcal{S}$ of points is a k-convex guard set for $\mathcal{P}$ if every convex k-gon formed from points of $\mathcal{P}$ contains a guard from $\mathcal{S}$.*

We consider the following problem.

Problem 1.1 *Let $k \geq 3$ be an integer. Given a set $\mathcal{P}$ of n points in general position in the plane what is the minimum size of a k-convex guard set for the point-set $\mathcal{P}$?*

This gives rise to the following definition.

Definition 1.2 *Let $k \geq 3$ be an integer. Define $G_k(n)$ to be the minimum positive integer s such that every planar point-set $\mathcal{P}$ of size n has a k-convex guard set of size s.*

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If we restrict attention to convex point-sets $\mathcal{C}$ then we have the following definition.

Definition 1.3 *Let $k \geq 3$ be an integer. Define $G_k^{\mathcal{C}}(n)$ to be the minimum positive integer s such that for every planar convex set $\mathcal{C}$ of size n has a k-convex guard set of size s.*

If we restrict attention to convex point-sets $\mathcal{R}$ whose points form a regular n-gon then we have the following definition.

Definition 1.4 *Let $k \geq 3$ be an integer. Define $G_k^{\mathcal{R}}(n)$ to be the minimum positive integer s such that for every set of points, which are vertices of a regular n-gon, has a k-convex guard set of size s.*

In the sequel we provide more precise upper and lower bounds for these functions.

It is clear from the definitions that

$$G_n(n) \leq G_{n-1}(n) \leq \cdots \leq G_3(n).$$

Similar inequalities hold for the functions $G_k^{\mathcal{C}}(n)$ and $G_k^{\mathcal{R}}(n)$.

2 Convex Point-sets

In this section we consider only convex point-sets. It is easy to see that from any set of n points in convex position we can construct at most $\lfloor \frac{n-2}{k-2} \rfloor$ pairwise non-overlapping convex k-gons. Therefore the following result follows easily.

Lemma 2.1 *For every $k \leq n$, every convex set of size n has a k-convex guard set of size $\lfloor \frac{n-2}{k-2} \rfloor$.* ■

Theorem 2.1 below is an immediate consequence of the special case in Theorem 2.3 when the point-set is convex, but we give a separate proof to aid our understanding.

Theorem 2.1 *For any $n \geq 3$, $C_3^{\mathcal{C}}(n) = n - 2$. i.e., for any convex set $\mathcal{C}$ of n points in general position in the plane $n - 2$ points are necessary and sufficient to guard all triangles formed by three points from $\mathcal{C}$.*

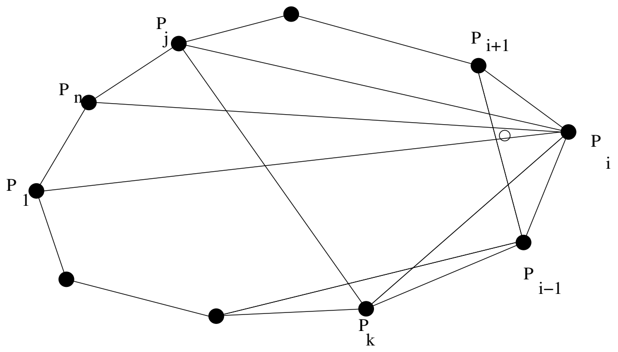

Figure 1: Proving the upper bound for convex point-sets.

PROOF The lower bound follows immediately from Lemma 2.1 when $k = 3$. To prove the upper bound consider a convex point-set $\mathcal{C}$ consisting of n vertices $P_1, P_2, \ldots, P_n$ in clockwise order. For each $2 \leq i \leq n-1$ put a guard P'_i inside the intersection of the interior of the triangles $P_1P_iP_n$ and $P_{i-1}P_iP_{i+1}$. Let $S = \{P'_2, P'_3, \ldots, P'_{n-1}\}$. We show that the interior of any triangle $P_iP_jP_k$, where $1 \leq k < i < j \leq n-1$, contains a guard from S. Consider the guard P'_j. By definition it lies at the intersection of the interiors of the triangles $P_1P_jP_n$ and $P_{j-1}P_jP_{j+1}$ and therefore also at the interior of the triangle $P_iP_jP_k$ (see Figure 1).

This completes the proof of the theorem. ∎

Using Theorem 2.1 and repeating the proof of Theorem 3.3 from the next section we obtain the following extention.

Theorem 2.2 *For any $n \geq 3k$, $C^{\mathcal{C}}_{2k+1}(n) \leq n - 2k$. i.e., for any convex set $\mathcal{C}$ of n points in general position in the plane $n - 2k$ points are sufficient to guard all convex $(2k+1)$-gons formed by k points from $\mathcal{C}$.* ∎

Lemma 2.1 and Theorem2.1 imply that

$$\left\lfloor \frac{n-2}{2k-1} \right\rfloor \leq C^{\mathcal{C}}_{2k+1}(n) = n - 2k,$$

but in general we do not know of any tighter bounds for convex sets. However, for regular n-gons we can prove the following result.

Theorem 2.3 *For any $n \geq k$, $\Omega(n/k) \leq G^{\mathcal{R}}_k(n) \leq O((n/k)^2)$, i.e., $\Omega(n/k)$-many k-convex guards are necessary and $O((n/k)^2)$-many k-convex guards are sufficient to guard all the k-convex subsets of a regular n-gon.*

PROOF (OUTLINE) The lower bound follows from Theorem 2.1. Let $\mathcal{P}$ be the set of vertices of a regular n-gon placed on a circle of radius 1. Notice that there exists a constant c, such that in the interior of any convex k-gon we can place a circle of radius c/k. This indicates that the following set of guards would be sufficient. Take $O(n/k)$ equally spaced diameters of the regular polygon and place $O(n/k)$ guards equally spaced on each such diameter. This proves the theorem. ∎

3 Arbitrary Point-sets

In the sequel we extend Theorem 2.1 to arbitrary point-sets. In particular we prove the following result.

Theorem 3.1 *Let n be an integer ≥ 3, For any set $\mathcal{P}$ of n points in general position in the plane, $(2n-5)$-many 3-convex guards are necessary and sufficient to guard every triangle formed by three points of $\mathcal{P}$.* ∎

In fact, since the convex hull of a point-set always has at least 3 points this is an immediate consequence of the following stronger result.

Theorem 3.2 *Let n be an integer ≥ 3, For any set $\mathcal{P}$ of n points in general position in the plane with c points in its convex hull, $(2n-c-2)$-many 3-convex guards are necessary and sufficient to guard every triangle formed by three points of $\mathcal{P}$.*

PROOF Let $\mathcal{P}$ be a point-set consisting of n points which has c vertices in its convex hull. First of all we prove that $(2n-c-2)$-many 3-convex guards are sufficient to guard every triangle formed by three points of $\mathcal{P}$. Take any point P in the convex hull of the point-set $\mathcal{P}$. With each other point $Q \in \mathcal{P} \setminus \{P\}$, if Q belongs to the convex hull of $\mathcal{P}$, we associate one guard, otherwise we associate two guards in the following way. Suppose first that Q is not a hull point. Take point $S \in \mathcal{P}$, left to the line defined by the segment PQ, such that the wedge PQS does not contain in its interior any other point from $\mathcal{P}$. We will call it the *lower Q-wedge*. Place a guard Q' in the intersection of lower Q-wedge with all the triangles containing point Q and having vertices at points of $\mathcal{P}$. In similar way place a guard Q'' in the *upper Q-wedge* (see Figure 2).

If Q is a hull vertex, we place a guard in the intersection of lower Q-wedge with all the triangles having one vertex at Q and the two remaining vertices lying on the different sides of line PQ. We will also permit that the vertex which have to lie right to PQ may be point P itself.

It remains to verify that every triangle built from the points of $\mathcal{P}$ contains one of $2n-c-2$ guards defined above. Take any such triangle and draw the line l passing through point P and one of its vertices, that we call Q, such that the remaining vertices are separated by l (see Figure 3 (a) and (b)). This is always possible, unless one of the vertices of the triangle is point P

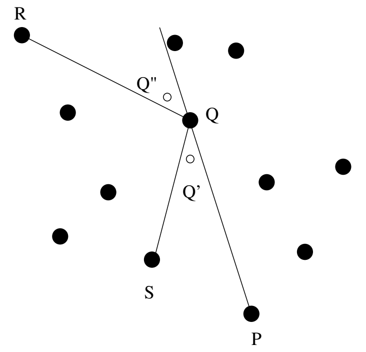

Figure 2: Proving the upper bound for arbitrary point-sets.

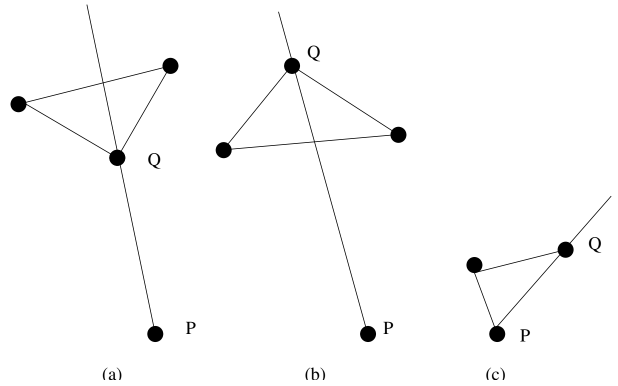

Figure 3: The three types of triangles with one vertex equal to Q.

itself. In this case we chose as vertex Q the one which will leave the triangle on the left-hand side of l (see Figure 3 (c)).

It is possible to show that each such triangle contains a guard associated with point Q. This guard is from the upper Q-wedge for the triangle of type (a) and from the lower Q-wedge for triangles of type (b) and (c). As for Q being the hull vertex only triangles type (b) and (c) are possible, the single guard placed at the lower Q-wedge was sufficient.

This completes the proof of our desired assertion and hence also the proof that $(2n - c - 2)$-many 3-convex guards are always sufficient for guarding point-sets with c points in their convex hull.

To prove that $2n-c-2$ points are also necessary, note that every triangulation of $\mathcal{P}$ has $2n - c - 2$ triangles. As each of them needs a guard this completes the proof of the theorem. ■

As a corollary we obtain the following result.

Corollary 3.1 *For any set $\mathcal{P}$ of n points in general position in the plane with c points in its convex hull there is a collection $\mathcal{S}$ of $n-2$ points such that for every triangulation of $\mathcal{P}$ every triangle from the triangulation contains exactly one point from $\mathcal{S}$ in its interior.* ■

Next we consider a bound for guarding convex k-gons when $k > 3$. More precisely we can prove the following theorem.

Theorem 3.3 *For any $n \geq 3k$, $G_{2k+1}(n) \leq 2n - 5k$, i.e., for any set $\mathcal{P}$ of n points in general position in the plane $(2n - 5k)$-many $(2k + 1)$-convex guards are always sufficient to guard every convex $(2k + 1)$-gon formed from points of $\mathcal{P}$.*

Proof Partition $\mathcal{P}$ into k pointsets $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ each of size at least ≥ 3. By Theorem 3.2, for each $i = 1, 2, \dots, k$, there is a set $\mathcal{S}_i$ of size at most $2|\mathcal{P}_i| - 5$ such that every triangle formed from points of $\mathcal{P}_i$ has a guard from $\mathcal{S}_i$ in its interior. Define the set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k$. Clearly,

$$\begin{aligned} |\mathcal{S}| &\leq \sum_{i=1}^{k} |\mathcal{S}_i| \\ &\leq \sum_{i=1}^{k} (2|\mathcal{P}_i| - 5) \\ &\leq 2n - 5k. \end{aligned} \tag{1}$$

Let C be a convex $(2k+1)$-gon formed from vertices of $\mathcal{P}$. There exists an integer i and three points P, Q, R from C such that $P, Q, R \in \mathcal{P}_i$. Indeed, otherwise for all $i = 1, \dots, k$, $|C \cap \mathcal{P}_i| \leq 2$, which implies that $2k + 1 = |C| = \sum_{i=1}^{k} |C \cap \mathcal{P}_i| \leq 2k$, which is a contradiction. However, the triangle PQR must have a guard in $\mathcal{S}_i$. It follows that C also has a guard in $\mathcal{S}_i$. Hence C has a guard in $\mathcal{S}$. It follows that $G_{2k+1}(n) \leq 2n - 5k$ and the proof of the theorem is complete. ■

A stronger upper bound can be proved when $n \geq k \log n$ by selecting appropriately the partition above.

Theorem 3.4 *For any $n \geq k \log n$, $G_{2k+1}(n) \leq$*

$$2(n - k) - 3 - \frac{k-1}{2} \log(n - (k-2) \log n),$$

i.e., for any set $\mathcal{P}$ of n points in general position in the plane $(2(n - k) - 3 - \frac{k-1}{2} \log(n - (k-2) \log n))$-many $(2k + 1)$-convex guards are always sufficient to guard every convex $(2k + 1)$-gon formed from points of $\mathcal{P}$.

Proof By the Erdös-Szekeres theorem [2], from every set of n points in the plane, with no three points collinear, one can always select the vertices of a convex $\lceil \log n \rceil$-gon. In the proof of Theorem 3.3 above we select the partition $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ of $\mathcal{P}$ as follows. Choose $\mathcal{P}_1$ to be a convex subset of $\mathcal{P}$ of size $c_1 \geq \log n$. Choose $\mathcal{P}_2$ to be a convex subset of $\mathcal{P} \setminus \mathcal{P}_1$ of size $c_2 \geq \log(n - \log n)$. Continue inductively and define $\mathcal{P}_{i+1}$ as a convex subset of $\mathcal{P} \setminus (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_i)$ of size $c_i \geq \log(n - i \log n)$, for $i + 1 < k$. The last set $\mathcal{P}_k$ may not be convex, but it will have a convex hull of size $c_k \geq 3$.

Since by theorem 3.2, for each $i = 1, 2, \ldots, k$

$$|\mathcal{S}_i| = 2|\mathcal{P}_i| - c_i - 2$$

repeating the proof of Theorem 3.3, Inequality 1 becomes

$$\begin{aligned} |\mathcal{S}| &\leq \sum_{i=1}^{k} |\mathcal{S}_i| \\ &\leq 2n - 2k - \sum_{i=1}^{k} c_i \\ &\leq 2n - 2k - 3 - \tfrac{k-1}{2} \log(n - (k-2)\log n). \end{aligned}$$

The rest of the proof remains the same. This completes the proof of the theorem. ■

4 Open Problems

It would be interesting to obtain tighter bounds for the function $G_k(n), G_k^{\mathcal{C}}(n), G_k^{\mathcal{R}}(n)$.

References

[1] H. T. Croft, K. J. Falconer, and R. K. Guy, "Unsolved Problems in Geometry", Problem Books in Mathematics, Vol. 11, Springer Verlag, 1991.

[2] P. Erdös and G. Szekeres, "A Combinatorial Problem in Geometry", Composito Math., 2, pp. 263-270, 1935.