On the Average Complexity of 3D-Voronoi Diagrams of Random Points on Convex Polytopes (Extended Abstract)

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Abstract

It is well known that the complexity, i.e., the number of vertices, edges and faces, of the 3-dimensional Voronoi diagram of n points can be as bad as $\Theta(n^2)$. It is also known that if the points are chosen Independently Identically Distributed uniformly from a 3-dimensional region such as a cube or sphere, then the *expected* complexity falls to $O(n)$. In this paper we analyze what happens if the points are chosen from a 2-dimensional region in 3-dimensional space. As an example, we examine the situation when the points are chosen from the *surface* of a convex polytope.

1 Introduction

Given a set $S_n = \{p_1, p_2, \ldots, p_n\} \subseteq \Re^d$ of n points in d-dimensional Euclidean space, the *Voronoi Diagram*, $\mathrm{VD}(S_n)$, of S_n is a very well understood subdivision of R^d. For each point $p_i \in S_n$ there is an associated (convex) cell

$$C_i = \{x \in \Re^d \,:\, \forall j \neq i,\, d(x,p_i) \leq d(x,p_j)\}$$

where $d(\cdot,\cdot)$ is the Euclidean distance function. By definition these cells partition space. The *complexity* of $VD(S_n)$ is the number of lower dimensional pieces that compose $\mathrm{VD}(S_n)$. For example, in the two-dimensional case, $d = 2$, $\mathrm{VD}(S_n)$ contains edges $\mathcal{E}$ and vertices $\mathcal{V}$. The complexity of $\mathrm{VD}(S_n)$ will be $|\mathrm{VD}(S_n)| = |\mathcal{E}| + |\mathcal{V}|$. Since it is also known that the two-dimensional Voronoi Diagram is a planar graph, Euler's theorem immediately implies that $|\mathrm{VD}(S_n)| = \Theta(n)$ [6].

If $d = 3$ then $\mathrm{VD}(S_n)$ is composed not only of the edges $\mathcal{E}$ and vertices $\mathcal{V}$ but also of the faces $\mathcal{F}$ of the convex cells. The complexity of $\mathrm{VD}(S_n)$ will then be $|\mathrm{VD}(S_n)| = |\mathcal{E}| + |\mathcal{V}| + |\mathcal{F}|$. In three dimensions it can be proven that $|\mathrm{VD}(S_n)| = O(n^2)$. For some cases, such as when all of the points in S_n are on the *moment curve* $\{(t,t^2,t^3) \,:\, t \in \Re\}$, it can be easily proven that $\mathcal{V} = \Theta(n^2)$ so $|\mathrm{VD}(S_n)| = \Theta(n^2)$ [5]. Another well known example of this worst-case behavior is built around two line segments $L_1 = \{(x,0,0) : x \in [0,\frac{1}{2}]\}$ and $L_2 = \{(1,y,1) : y \in [\frac{1}{2},1]\}$; given any n_1 points on L_1 and any n_2 points on L_2, $|\mathcal{V}| = \Theta(n_1 n_2)$. In particular if $n_1 = n_2 = \frac{n}{2}$ then $|\mathcal{V}| = \Theta(n^2)$.

Moving away from worst-case behavior to average-case behavior it has been shown that if n points of S_n are independently identically distributed (IID) chosen from the uniform distribution over a "reasonably" smooth full dimensional bounded region $\mathcal{P}$ such as a cube or sphere then $E(|\mathrm{VD}(S_n)|) = \Theta(n)$ [1][3][4]. [1].

[1]These references don't exactly state this

Dropping the condition that $\mathcal{P}$ has full dimensionality dramatically changes the situation. For example, if we set $P = L_1 \cup L_2$ to be the union of the two 1-dimensional segments previously defined and choose n points S_n randomly from P then $n_1 = |S_n \cap L_1|$, the number of points on L_1, is a binomial random variable with parameters $n, \frac{1}{2}$. Since $n_2 = |S_n \cap L_2| = n - n_1$ and $\mathcal{V} = \Theta(n_1 n_2)$ we have that

$$E(|\mathcal{V}|) = \Theta(E(n_1(n-n_1))) = \Theta(n^2)$$

and $E(|\mathrm{VD}(S_n)|) = \Theta(n^2)$.

Combining the two previous paragraphs we see that, in 3-dimensional space, if n points S_n are chosen IID uniformly from $\mathcal{P}$ where $\mathcal{P}$ is a reasonably smooth 3-dimensional region then $E(|\mathrm{VD}(S_n)|) = \Theta(n)$ while for some 1-dimensional $\mathcal{P}$s, $E(|\mathrm{VD}(S_n)|) = \Theta(n^2)$. The obvious question then is what happens if $\mathcal{P}$ is a *2-dimensional* surface in 3-dimensional space and n points S_n are chosen IID uniformly from it. What will be the expected complexity $E(|\mathrm{VD}(S_n)|)$ of the 3-dimensional Voronoi diagram of those points? $\Theta(n^2)$? $\Theta(n)$? Something in between?

In this note we make a first step towards answering this question. We prove

Theorem 1 *Let $\mathcal{P}$ be the boundary of a convex polytope in $\Re^3$. Let S_n be a set of points chosen from the standard 2-dimensional Poisson distribution on $\mathcal{P}$ with rate n. Then $E(|\mathrm{VD}(S_n)|) = \Theta(n)$.*

The Poisson distribution on $\mathcal{P}$ with rate n is the one that has the properties: (i) let $M \subseteq \mathcal{P}$ be any measurable region and $N(M)$ be the random variable signifying the number of points the process generates in M (the dependence upon n is implicit). Then

$$\Pr(N(M) = k) = \frac{(n\mathrm{Area}(M))^k e^{-(n\mathrm{Area}(M))}}{k!} \tag{1}$$

(so $E(N(M)) = n\mathrm{Area}(M)$). Furthermore, (ii) if M_1 and M_2 are nonoverlapping regions, then $N(M_1)$ and $N(M_2)$ are independent random variables. We note that we have restricted ourselves to proving Theorem 1 for a Poisson distribution because its mathematics are a bit cleaner (it allows us to assume that points in various regions are chosen independently of each other) but standard modifications allow the proof to also work for n points chosen IID uniformly from $\mathcal{P}$ and show that, in this case as well, $E(|\mathrm{VD}(S_n)|) = \Theta(n)$.

Note: In what follows we use $S(p,r) = \{q \in \Re^3 : d(q,p) \le r\}$ to denote the 3-dimensional sphere of radius r around p and $B_\Pi(p,r) = \{q \in \Pi : d(q,p) \le r\}$ to denote the 2-dimensional ball of radius r around p on plane Π; if Π is implicit then we will just write $B(p,r)$.

For point p and set S the notation $NN(p,S)$ will denote a nearest neighbor q to p in S, i.e. a $q \in S$ such that

$$\forall q' \in S,\ d(p,q) \le d(p,q').$$

In this paper all of the sets S used will either be finite or closed. Thus such a q will always exist although it might not always be unique.

We also define the *Skeleton* and *r-boundary* of $\mathcal{P}$:

$$\begin{aligned} Skel(\mathcal{P}) &= \{u \in \mathcal{P} : u \text{ is on some edge of } \mathcal{P}\} \\ Bd(r) &= \{u \in \mathcal{P} : \exists \text{point } v \in Skel(\mathcal{P}) \text{ such that } d(u,v) < r\}. \end{aligned}$$

Thus $Bd(r)$ is the set of points on $\mathcal{P}$ within distance r of an edge or vertex of $\mathcal{P}$.

Finally we point out that in this extended abstract we only sketch the proof of Theorem 1 but do not provide full details. Those will be given in the journal version of this work.

fact but it can be inferred from the general techniques developed there. The idea behind the proof is that points p_i *inside* $\mathcal{P}$ only have a constant number of Voronoi *neighbors* so their Voronoi cells C_i will have constant complexity. Points near the boundary of $\mathcal{P}$ might have Voronoi cells with high complexity but there are only a small number of such boundary cells.

2 A Sketch of the Proof

For a point $p \in \Re^3$, let $r_p = \min\{d(p,p') : p' \in S_n\}$ and set $S = S(p, r_p)$ to be the *defining sphere* around p. S contains no points of S_n in its interior and at least one point of S_n on its boundary. We will call the points of S_n on S's boundary the *defining* points of S. It is well known that every vertex/edge/face of $\mathrm{VD}(S_n)$ corresponds to at least one defining sphere with $4/3/2$ defining points of S_n on its boundary. Our strategy for bounding the complexity of $\mathrm{VD}(S_n)$ will therefore be *to bound the number of different possible defining spheres S* where two spheres are considered to be different only if they have different defining points. Note that we will use the terms *defining spheres* and *Voronoi spheres* interchangeably.

In this extended abstract we only provide a sketch of the proof. We start by introducing the concept of a *good*-sphere and then showing that we can *almost* restrict ourselves to enumerating just good spheres.

Definition 1 *Let F be a convex polygon on plane Π, v a vertex of F, e one of the two edges incident upon v and $\mathcal{G}$ an infinite square lattice gridding of Π oriented such that*

1. *the origin of $\mathcal{G}$ is v.*
2. *the horizontal and vertical spacings of $\mathcal{G}$ are both $1/n$.*
3. *e lies on one of the two lattice lines through v.*

*A triangle $T \subseteq \Pi$ is x-*bad with respect to *$\mathcal{G}$ if all three of the vertices of T are lattice points in $\mathcal{G}$ that fall in F, and $Area(T) \geq x^2$.*

*A triangle $T \subseteq \Pi$ is x-*bad with respect to *F if $\exists \mathcal{G}$ that satisfies the three conditions given above and T is x-*bad with respect to *$\mathcal{G}$.*

*Now let $\mathcal{P}$ be a polytope, $F_1, F_2, \ldots, F_k$ its faces and $\Pi_1, \Pi_2, \ldots, \Pi_k$ their corresponding supporting planes. A sphere S in $\Re^3$ is x-*bad *if $\exists F_i$ and triangle $T \subseteq \Pi_i$ such that T is x-*bad with respect to *F_i and $T \subseteq F_i \cap S$.*

*A sphere S in $\Re^3$ is x-*good *if it is not x-*bad.

The reason for introducing this definition is that while there are an uncountable number of spheres in $\Re^3$ and therefore an uncountable number of Voronoi sphere candidates there are only a polynomial number of possible bad triangles. In fact, for any face F_i and *fixed* grid $\mathcal{G}$ the total number of triangles that are x-bad with respect to $\mathcal{G}$ is at most the number of possible triples of grid points which is $O(n^6)$, where the constant in the $O()$ notation is only dependent upon the lengths of the edges of F_i. At most $|E_i|$ possible square grids $\mathcal{G}$ can be defined in Definition 1 for face F_i, where $|E_i|$ is the total number of edges on face F_i. Counting over all faces we find that the total number of grids possible in the definition for $\mathcal{P}$ is $2|E|$ where $|E|$ is the total number of edges in $\mathcal{P}$. Thus, the total number of triangles that are bad with respect to some face is $O(|E|n^6) = O(n^6)$ where the constant in the $O()$ is dependent only upon $\mathcal{P}$. Now notice that the probability that any given x-bad triangle T contains no point of S_n is $\Pr(|S_n \cap T| = 0) = e^{-n\,Area(T)} \leq e^{-nx^2}$ by the definition of the Poisson distribution (see (1)). The probability that there exist some T and F such that T is *x-bad with respect to F* and $T \cap S_n = \emptyset$ is then $O(n^6 e^{-nx^2})$. Hence the probability that there exists a x-bad sphere S such that $S \cap S_n = \emptyset$ is then also the same $O(n^6 e^{-nx^2})$. Plugging the value $x = \frac{\log n}{\sqrt{n}}$ into this argument proves

Lemma 1 *Let S_n be a set of points chosen from the standard 2-dimensional Poisson distribution on $\mathcal{P}$ with rate n. Then*

$$\Pr(\textit{there exists a } \tfrac{\log n}{\sqrt{n}}\text{-bad } \textit{sphere } S \textit{ such that } S_n \cap S = \emptyset) = n^{-\Omega(\log n)}.$$

The following two lemmas and associated corollary will be useful.

Lemma 2 *Let F be a convex polygon and $T \subseteq F$ a triangle such that $Area(T) \geq \frac{\log^2 n}{n}$. If n is large enough ('large enough' depending only upon F and not upon T,) $\exists T' \subseteq T$, such that triangle T' is $\frac{\log n}{\sqrt{n}}$*-bad with respect to *F.*

This lemma follows from the fact that if the area of T is large compared to that of a grid square then T must contain a large triangle T' whose vertices are grid points. We also have the immediate corollary:

Corollary 3 *Let S be a sphere such that $\exists F_i, T$ with $T \subseteq S \cap F_i$. For large enough n depending only upon $\mathcal{P}$, S is* $\frac{\log n}{\sqrt{n}}$-bad.

Lemma 4 *There exists $c > 0$ dependent only upon $\mathcal{P}$ with the following property: let $S = S(p,r)$ be a sphere, F be a face of $\mathcal{P}$, Π its supporting plane and $B = S(p,r) \cap \Pi$ be the intersection circle of S with Π. Now let $p' = NN(p, \Pi)$ and set r' to be such that $B = B_\Pi(p', r')$. Then $p' \in F$ and $r' \geq c\frac{\log n}{\sqrt{n}}$ together imply that $S \cap F$ contains a triangle T that is $\frac{\log n}{\sqrt{n}}$-bad with respect to F so S is* $\frac{\log n}{\sqrt{n}}$-bad.

We need one more set of definitions.

Definition 2 *Let $\mathcal{P}$ be a convex polytope and $S = S(p,r)$ a sphere.*

S is a Type-I *sphere if $\exists$ a face F of $\mathcal{P}$ such that $S \cap \mathcal{P} \subseteq F$.*

S is a Type-II *sphere if (i) it is not a Type-I sphere, (ii) p is* inside *$\mathcal{P}$ and (iii) for every face F_i of $\mathcal{P}$ with corresponding supporting plane Π_i, if $S \cap F_i \neq \emptyset$ then $NN(p, \Pi_i) \in F_i$.*

If S is not a Type-I or Type-II sphere then S is a Type-III *sphere.*

Type-I spheres are easily understood. Intuitively if p is inside $\mathcal{P}$ and S is not a type-I sphere, then S is a Type-II sphere if, for every face F_i of $\mathcal{P}$ that S intersects, the center of the circle formed by $S \cap \Pi_i$ is in F_i.

We will now sketch the proof technique; it is to count spheres of the form $S(p,r)$ where the boundary of $S(p,r)$ contains at least two points of S_n. If the interior of $S(p,r)$ is empty it will be a Voronoi sphere. Two spheres will be equivalent, if they have the same point(s) on their boundary. Then

$$
\begin{aligned}
&\text{No. of Voronoi Spheres}\\
&= \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Bad Voronoi Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Voronoi Spheres}\\
&= \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Bad V. Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-I V. Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-II V. Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-III V. Spheres}\\
&\leq \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Bad V. Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-I V. Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-II V. Spheres}\\
&\quad + \text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-III Spheres}
\end{aligned}
$$

We will then show that

$$E(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Bad V. Spheres}) = o(1). \quad (2)$$

$$E(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-I V. Spheres}) = O(n). \quad (3)$$

$$E(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-II V. Spheres}) = O(n). \quad (4)$$

$$E(\text{No. of } \tfrac{\log n}{\sqrt{n}}\text{-Good Type-III Spheres}) = o(n). \quad (5)$$

Taking expectations of the left and right hand sides of the previous inequality together, the linearity of the expectation operator will finally imply our Theorem 1, that the expected number of Voronoi spheres, which is the same as the expected complexity of the Voronoi Diagram, will be $O(n)$. The reason that we use this decomposition is that it permits us to upper bound the number of Good Type-III *Voronoi* Spheres by the number of Good Type-III Spheres. Bounding the number of the Voronoi Spheres would be quite difficult since assuming that a sphere is Voronoi, i.e., empty, requires conditioning on skewing the rest of the distribution of Voronoi Spheres. Bounding the number of Good Spheres is much easier since it only requires calculating how many points could feasibly fall within a particular volume or area.

Let $A = \sum_i Area(F_i)$ be the total surface area of $\mathcal{P}$. Proving (3) is actually quite straightforward. It follows almost directly from Lemma 1, the fact that $\Pr(|S_n| > 2An) = n^{-\Omega(\log n)}$ (a very conservative bound) and that point set S_n defines at most $O(|S_n|^2)$ Voronoi spheres.

Proving (4) is also quite easy. From the definition of Type-II spheres we have that $S_n \cap S \subseteq F$ for some face F of $\mathcal{P}$. This means that S will be a defining sphere of the *planar* Voronoi diagram of the points $S_n \cap S$ on the supporting plane Π of F. Since planar Voronoi diagrams of m points have complexity $O(m)$ we immediately have that the total number of Type-I Voronoi Spheres (we no longer have to restrict ourselves to good ones) corresponding to S intersecting face F_i is $O(|S_n \cap F_i|)$. Summing over all faces F_i gives that the total number of Type-I Voronoi Spheres is $O(\sum_i |S_n \cap F_i|) = O(|S_n|)$. The expected number of Type-I Voronoi Spheres is therefore $O(E(|S_n|)) = O(An) = O(n)$ proving (4).

We will prove (5) in the next section using a straightforward derivation.

The hardest part of the paper will be the proof of (5) which will require a tedious case by case analysis, examining the cases where p is *outside or on* $\mathcal{P}$ and where p is *inside* $\mathcal{P}$ separately and upper bounding the number of possible Type III good spheres for each case.

3 Bounding the Number of $\frac{\log n}{\sqrt{n}}$-Good Type-II Voronoi Spheres

In this section we bound the expected number of Good Type-II Voronoi Spheres. Our major tool is the following Lemma.

Lemma 5 *Let F_1, F_2, F_3, F_4 four convex 2-dimensional* polygons *in $\Re^3$ and $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ be their supporting planes. Further let $S_{F_1,F_2,F_3,F_4,n}$ be the set of points generated by the Poisson distribution with rate n on $\cup_{i=1}^4 F_i$. Set $X_{F_1,F_2,F_3,F_4,n}$ to be the number of Voronoi spheres $S = S(p,r)$ for the set $S_{F_1,F_2,F_3,F_4,n}$ such that $\forall i \le 4$, $NN(p,F_i) \in F_i$. Then $E(X_{F_1,F_2,F_3,F_4,n}) = O(n)$.*

Now suppose that S_n is the the set of points generated by the Poisson distribution with rate n on $\mathcal{P}$. If a sphere $S = S(p,r)$ is a Voronoi sphere there must be 4 faces $F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4}$ such that the defining points of S are on $F_{i_1} \cup F_{i_2} \cup F_{i_3} \cup F_{i_4}$. Furthermore, those points define a Voronoi sphere for the set $S_n \cap (F_{i_1} \cup F_{i_2} \cup F_{i_3} \cup F_{i_4})$ (the converse is not necessarily true). The distribution of $S_n \cap (F_{i_1} \cup F_{i_2} \cup F_{i_3} \cup F_{i_4})$ is the same as the distribution of $S_{F_{i_1},F_{i_2},F_{i_3},F_{i_4},n}$. Thus the expected number of Voronoi Good Spheres defined by S_n that satisfy the condition $\forall j \le 4$, $NN(p, F_{i_j}) \in F_{i_j}$ is *upper bounded* by $E(X_{F_{i_1},F_{i_2},F_{i_3},F_{i_4},n}) = O(n)$. Since there are at most (k^4) possible 4-tuples of faces, where k is the number of faces, this proves that the expected number of Type-II Voronoi Speres is $O(k^4 n) = O(n)$ demonstrating (5).

We note that the proof of Lemma 5 doesn't actually use the fact that the spheres are $\frac{\log n}{\sqrt{n}}$-Good.

4 Bounding the Number of $\frac{\log n}{\sqrt{n}}$-Good Type-III Spheres

As mentioned earlier this is the tedious portion of the paper. It requires a case-by-case analysis of all $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres $S(p,r)$. The upper level distinction of the cases is between those counting spheres with centers outside of $\mathcal{P}$ and those counting spheres with centers inside $\mathcal{P}$.

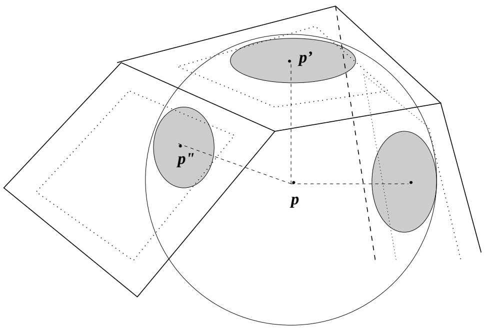

Figure 1: An example of a Type-II sphere. p, the center of the sphere, is inside polytope $\mathcal{P}$ (for clarity many of the faces of $\mathcal{P}$ have been left out of the diagram). For all the faces F_i of $\mathcal{P}$ that the sphere intersects, the center of the intersection circle with the supporting plane Π_i is on F_i.

4.1 Sphere center p *outside or on* $\mathcal{P}$

If p is *outside or on* $\mathcal{P}$, let $p' = NN(p, \mathcal{P})$. The convexity of $\mathcal{P}$ guarantees that p' is unique. Let $S = S(p, r)$ be a $\frac{\log n}{\sqrt{n}}$-Good Type-III sphere that is defined by 2/3/4 points of S_n on its boundary (but $S \cap S_n$ not necessarily being empty). There are three cases. These cases will be defined using parameters $c_1, c_2 > 0$ that are constants dependent only upon $\mathcal{P}$. The values of these constants are complicates and derived in the proofs.

Case1 $p' \notin Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$

Case2 $p' \in Bd(\frac{c_1 \log n}{\sqrt{n}})$ but $p' \notin Skel(\mathcal{P})$.

Case3 $p' \in Skel(\mathcal{P})$.

We now work through the cases.

Case1. If $p' \notin Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ we will see that $S(p, r)$ must be a Type-I sphere and should therefore not be counted here. Let F be the face of $\mathcal{P}$ such that $p' \in F$, and Π its supporting plane. Note that the circle $B = S \cap \Pi$ has center $p' \in F$, The constant c_1 is chosen so that if S intersects any other face of $\mathcal{P}$ then the radius r' of

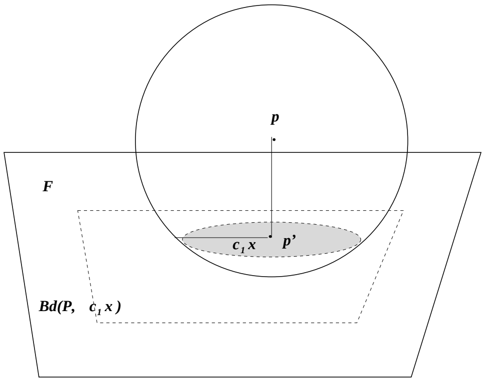

Figure 2: Case 1 for $\frac{\log n}{\sqrt{n}}$-Good Type-III Spheres with p outside or on $\mathcal{P}$. In this case notice that the intersection of S with $\mathcal{P}$ is a circle completely contained in F. If the intersection contains a part of some other face then the intersection circle on F would have to grow to be so big that S would no longer be $\frac{\log n}{\sqrt{n}}$-Good. This is a consequence of p' being at least distance $c_1 \frac{\log n}{\sqrt{n}}$ from the border.

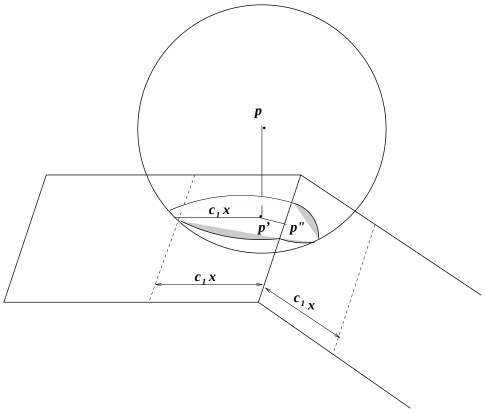

Figure 3: Case 2 for $\frac{\log n}{\sqrt{n}}$-Good Type-III Spheres with p outside or on $\mathcal{P}$. In this case notice that the center of the intersection circle of S with the supporting plane of the second face, the one not p', is not on the face. $\mathcal{P}$ is close to some point p'' on the boundary.

B satisfies $r' > c\frac{\log n}{\sqrt{n}}$ where c is the constant from Lemma 4, so S is a bad sphere. Since we are only counting good spheres this immediately implies that $S \cap \mathcal{P} \subseteq F$ so S is Type-I sphere and need not be examined here.

Case2. We need the following lemma which can be proven:

Lemma 6 *If $p' \in Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ but $p' \notin Skel(\mathcal{P})$, then $\exists q$ on some edge of $\mathcal{P}$, q dependent upon p, such that if $u \in S \cap \mathcal{P}$ then $d(u,q) \leq c_2\frac{\log n}{\sqrt{n}}$.*

We can use this lemma to show that the expected total number of $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres $S(p,r)$ with $p' \in Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ but $p' \notin Skel(\mathcal{P})$ is $O(n)$:

Let L be the total length of all of the edges of $\mathcal{P}$. We partition up the edges of P into $O(L/\frac{\log n}{\sqrt{n}})$ line segments each of length $\leq \frac{\log n}{\sqrt{n}}$, e.g., all segments will have length $\frac{\log n}{\sqrt{n}}$ except for, possibly, one segment per edge (which contains one of that edge's endpoints). Let s be any edge segment and $N(s) = \{u \in \mathcal{P} : d(u,s) \leq c_2\frac{\log n}{\sqrt{n}}\}$ be the set of all points on $\mathcal{P}$ within distance $\frac{\log n}{\sqrt{n}}$ of s. Then $Area(N(s)) \leq c_3(\frac{\log n}{\sqrt{n}})^2 = c_3\frac{\log^2 n}{n}$ for some constant c_3 dependent only upon $\mathcal{P}$. Let $X(s) = |N(s) \cap S_n|$. Plugging into the Poisson distribution we find that $E(X(s)) = O(\log^2 n)$, $E(X^2(s)) = O(\log^4 n)$, $E(X^3(s)) = O(\log^6 n)$, and $E(X^4(s)) = O(\log^8 n)$. For any sphere center p let q be the corresponding edge point defined by Lemma 6 and s the segment that q belongs to. From Lemma 6 and the definition of $N(s)$, the 2/3/4 points of S_n that define $S(p,r)$ must be in $N(s)$ and as we have already seen there are only $O(\log^8 n)$ number of such expected tuples of points *possible in* $N(s)$. The total expected number of spheres of this type is upper bounded by the number of such segments times the expected number of spheres that can correspond to q's in the segment. That is, $O(L/\frac{\log n}{\sqrt{n}}) \times O(\log^8 n) = O(\sqrt{n}\log^6 n) = O(n)$. This shows that the expected total number of $\frac{\log n}{\sqrt{n}}$-Good Type-II spheres $S(p,r)$ with $p' \in Bd\left(\frac{c_1 \log n}{\sqrt{n}}\right)$ but $p' \notin Skel(\mathcal{P})$ is $O(n)$.

Case3. We start by introducing two new definitions:

Definition 3 *For $p' \in Skel(\mathcal{P})$ define*

$M(p') = \{u \in \mathcal{P} : u \in S(p,r)$ for some $\frac{\log n}{\sqrt{n}}$-Good Type-III sphere $S(p,r)$ such that $p' = NN(p,\mathcal{P})\}$

For s a segment of an edge in $Skel(\mathcal{P})$

$$M(s) = \cup_{p' \in s} M(p').$$

The following lemma can be proven using basic geometrical arguments:

Lemma 7 *Let s be a segment of an edge in $Skel(\mathcal{P})$ with $length(s) \leq \frac{\log n}{\sqrt{n}}$. Then*

$$Area(M(s)) \leq c_4\frac{\log^3 n}{n}$$

for some c_4 dependent only upon $\mathcal{P}$.

To bound the expected number of $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres $S(p,r)$ with $p' \in Skel(\mathcal{P})$, we then proceed in a similar fashion as we did in Case 2. Let L be the total length of all of the edges of $\mathcal{P}$. We partition up the edges of P into $O(L/\frac{\log n}{\sqrt{n}})$ line segments each of length $\leq \frac{\log n}{\sqrt{n}}$, e.g., all segments will have length $\frac{\log n}{\sqrt{n}}$ except for, possibly, one segment per edge (which contains one of that edge's endpoints).

Now suppose $s \subseteq Skel(\mathcal{P})$ with $length(s) \leq \frac{\log n}{\sqrt{n}}$. Let $X(s) = |M(s) \cap S_n|$. Plugging into the Poisson distribution we find that $E(X(s)) = O(\log^3 n)$, $E(X^2(s)) = O(\log^6 n)$, $E(X^3(s)) = O(\log^9 n)$, and $E(X^4(s)) = O(\log^{12} n)$. For a sphere $S(p,r)$ of the type we are examining, let $p' = NN(p,\mathcal{P}) \in Skep(\mathcal{P})$. From Lemma 7 and the definition of $M(s)$, the 2/3/4 points of S_n that define $S(p,r)$ must be in $M(s)$ and as we have already seen there are only $O(\log^{12} n)$ number of such expected tuples of points *possible in* $M(s)$. The total expected number of spheres of this type is upper bounded by the number of such segments times the expected number of spheres that can correspond to p''s in the segment. That is, $O(L/\frac{\log n}{\sqrt{n}}) \times O(\log^{12} n) = O(\sqrt{n}\log^{11} n) = O(n)$. This shows that the expected total number of $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres $S(p,r)$ with $p' \in Skel(\mathcal{P})$ is $O(n)$.

Combining parts of Case 1, Case 2, and Case 3, we see that the expected total number of $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres $S(p,r)$ with p *outside or on* $\mathcal{P}$ is $O(n)$.

4.2 Sphere center p *inside* $\mathcal{P}$

The only case of $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres $S = S(p,r)$ that we have not yet addressed is when p is *inside* $\mathcal{P}$. In this case, from the definition of Type-III spheres, we know that $\exists$ a face F of $\mathcal{P}$, such that $F \cap S \neq \emptyset$ but $p' = NN(p,\Pi)$ is not in F, where Π is the supporting plane of F.

The main lemma we prove here is

Lemma 8 *Let* $S = S(p,r)$ *be a* $\frac{\log n}{\sqrt{n}}$-*Good Type-III sphere with* p *inside* $\mathcal{P}$. *If, for some face* F *of* $\mathcal{P}$, $F \cap S \neq \emptyset$ *but* $p' = NN(p,\Pi)$ *is not in* F, *where* Π *is the supporting plane of* F *then* $\exists q$ *on the boundary of* F *such that* $\forall u \in S(p,r)$, $d(q,u) \leq c_5 \frac{\log n}{\sqrt{n}}$ *where* c_5 *is dependent only upon* $\mathcal{P}$.

The proof of this lemma involves showing that if $S(p,r)$ satisfies the given conditions then r is actually quite small so the distance from p to Π is some small multiple of $\frac{\log n}{\sqrt{n}}$. This in turn implies that the distance between q, the nearest point to p on F, and any point in $S(p,r)$ is $\leq c_5 \frac{\log n}{\sqrt{n}}$.

We now use exactly the same analysis as that which was employed for Case 2 in the previous subsection (note the similarity of Lemma 8 to Lemma 6) to

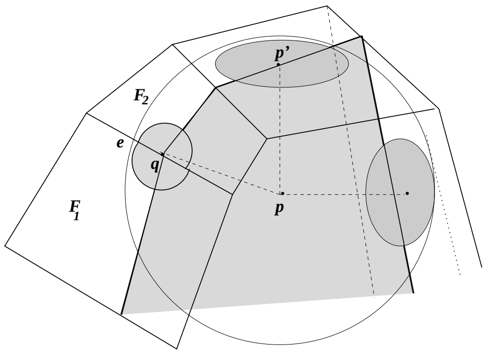

Figure 4: An example of a Type-III sphere $S = S(p,r)$ with p *inside* $\mathcal{P}$. Note that the nearest neighbor of p to F_1 is q which is on the boundary of F_1. The line segment pq is not perpendicular to F_1 which implies that the center of the intersection circle of S with the supporting plane of F_1 is not in F_1.

show that the expected number of $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres $S(p,r)$ with p *inside* $\mathcal{P}$ is $O(n)$. Combined with the results of the previous subsection, this shows that the total expected number of $\frac{\log n}{\sqrt{n}}$-Good Type-III spheres is $O(n)$ and we are done.

`Acknowledgment:` The authors would like to thank Herbert Edelsbrunner who suggested the original line of inquiry that led to this problem. They would also like to thank Ahn Hee-Kap, Sunil Arya, Siu-Wing Cheng, Otfried Cheong, Shin Chan-Su and Antoine Vigneron for various conversations concerning this problem.

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