On a straight-line embedding problem of forests

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Extended Abstract

We consider finite planar graphs without loops or multiple edges. Let G be a planar graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G, that is, $|G| = |V(G)|$. Given a planar graph G, let P be a set of $|G|$ points in the plane (2-dimentional Euclidean space) in general position (i.e., no three of them are collinear). Then G is said to be *line embedded onto* P or *stright-line embedded onto* P if G can be embedded in the plane so that every vertex of G corresponds to a point of P, every edge corresponds to a straight-line segment, and no two straight-line segments intersect except their common end-point. Namely, G is line embedded onto P if there exists a bijection $\phi : V(G) \to P$ such that two points $\phi(x)$ and $\phi(y)$ are joined by a straight-line segment if and only if x and y are joined by an edge of G and all two distinct open straight-line segments have no point in common. We call such a bijection a *line embedding* or a *straight-line embedding* of G onto P.

In this paper we consider line embeddings having one more property. Let G be a planar graph with n specified viertices $v_1, v_2, \ldots, v_n$, and P a set of $|G|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Then we say that $(G; v_1, v_2, \cdots, v_n)$ is *strongly line embedded onto* $(P; p_1, p_2, \cdots, p_n)$ if G can be line embedded onto P so that for every $1 \leq i \leq n$, v_i corrresponds to p_i, that is, if there exists a line embedding $\phi : V(G) \to P$ such that $\phi(v_i) = p_i$ for all $1 \leq i \leq n$. The line embedding mentioned above is called a *strong line embedding* of $(G; v_1, v_2, \cdots, v_n)$ onto $(P; p_1, p_2, \cdots, p_n)$. For a tree T and $v \in V(T)$, the pair

$(T; v)$ is called a *rooted tree*. For a forest F consisting of n trees $T_1, T_2, \cdots, T_n$ and $v_1, v_2, \cdots, v_n$ such that $v_i \in V(T_i)$ for each i, $(F; v_1, v_2, \cdots, v_n)$ is said to be a *rooted forest* consisting of n rooted trees.

We begin with the following theorem, which was conjectured by Perles [6] and partially solved by Pach and Törőcsik [5]; a simpler proof can be found in Tokunaga [8]. Another related result can be found in [1].

Theorem A ([2]) *Let T be a tree and v be a vertex of T. Then, for every set P of $|T|$ points in the plane in general position and each choice of $p \in P$, a rooted tree $(T; p)$ can be strongly line embedded onto $(P; p)$.*

Kaneko and Kano [3] extended the problem as follows:

Theorem B ([3]) *Let F be a forest consisting of two trees, and v_1, v_2 be vertices of distinct trees of F. Then for every set P of $|F|$ points in the plane in general position and each choice of distinct two points $p_1, p_2 \in P$, the rooted forest $(F; v_1, v_2)$ can be strongly line embedded onto $(P; p_1, p_2)$.*

It is easy to see that there exist rooted forests consisting of four rooted trees which cannot be strongly line embedded onto certain sets of points in the plane in general position containing four specified points. Thus, 'three' is the the bound. Kaneko and Kano had conjectured that 'two' in Theorem B can be incresed to 'three'. In the previous paper[4], we show the counterexample of this conjecture. We then introduce the definition of *semistrong line enbedding* by weakening the condition of strong line enbedding. $(G; v_1, v_2, \cdots, v_n)$ is said to be *semistrongly line embedded onto* $(P; p_1, p_2, \cdots, p_n)$ if there exists a line embedding $\phi : V(G) \to P$ such that $\phi(\{v_1, v_2, \cdots, v_n\}) = \{p_1, p_2, \cdots, p_n\}$.

In [4], we also prove the following theorem:

Theorem C ([4]) *Let F be a forest consisting of three trees, and v_1, v_2, v_3 be vertices of distinct trees of F. Then for every set P of $|F|$ points in the plane in general position and $p_1, p_2, p_3 \in P$, the rooted forest $(F; v_1, v_2, v_3)$ can be semistrongly line embedded onto $(P; p_1, p_2, v_3)$.*

We also conjectured that the similar statement holds for rooted forests consisting of more than three rooted trees. In this paper, as a partial solution of the conjecture, we prove the following theorem:

Theorem 1 *Let F be a forest consisting of n trees, and $v_1, v_2, \cdots, v_n$ be vertices of distinct trees of F. Let P be a set of $|F|$ points in the plane in general position and $p_1, p_2, \cdots, p_n$ be points of P in convex position. Then, the rooted forest $(F; v_1, \cdots, v_n)$ can be semistrongly line embedded onto $(P; p_1, \cdots, p_n)$.*

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