# Minimal Tetrahedralizations of a Class of Polyhedra *

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**Abstract**

Given an ordinary polyhedron $P$ in the three dimensional Euclidean space, different tetrahedralizations of $P$ may contain different numbers of tetrahedra. Minimal tetrahedralization is a tetrahedralization with the minimum number of tetrahedra. In this paper, we present some properties of the graph of polyhedra. Then we identify a class of polyhedra and show that this kind of polyhedra can be minimally tetrahedralized in $O(n^2)$ time.

**Keywords**: Computational geometry; Polyhedron; Tetrahedralization.

## 1 Introduction

The main purpose of decomposing an object into simpler components is to simplify a problem for complex objects into a number of subproblems for simpler components. In particular, a *tetrahedralization* is an edge set that partitions the input domain in $R^3$ (i.e., three dimensional Euclidean space) into a number of tetrahedra that meet only at shared faces. The tetrahedralization has applications in the finite element method, mesh generation, CAD/CAM, computer graphics, and robotics [1, 3, 4, 10, 13].

All convex polyhedra are tetrahedralizable, but not all nonconvex polyhedra can be tetrahedralized [12, 16]. If additional vertices, so-called Steiner points, are allowed when constructing the tetrahedralization, then all polyhedra are tetrahedralizable. Chazelle [5] showed that an ordinary polyhedron of $n$ vertices can always be partitioned into $O(n^2)$ convex regions, if Steiner points are allowed. This bound is tight in the worst case. Clearly, this is also a lower bound for tetrahedralization in the worst case. Chazelle and Palios [6] developed a tetrahedralization algorithm for decomposing an ordinary polyhedron with $n$ vertices and $r$ reflex edges into $O(n+r^2)$ tetrahedra using $O(n+r^2)$ Steiner points. Dey [9] and Hershberger and Snoeyink [11] presented other algorithms for tetrahedralization of polyhedra with Steiner points. Chazelle and Shouraboura [7] studied the polyhedron that is the region between two polyhedra. For this special kind of polyhedra, their algorithm produces $O(n)$ tetrahedra by using Steiner points. Ruppert and Seidel [15] proved that the problem of deciding whether a given polyhedron can be tetrahedralized without Steiner points is NP-complete; the problem remains NP-complete even for star-shaped polyhedra. Various versions of the question of how many Steiner points are needed to tetrahedralize a polyhedron also turn out to be NP-complete.

For the optimal tetrahedralization without Steiner points, very few results are known. Since different tetrahedralizations of the same polyhedron may contain different numbers of tetrahedra, a natural optimization problem is how to tetrahedralize a polyhedron with the minimum number of tetrahedra. The tetrahedralization with minimum number of tetrahedra is referred to as *minimal tetrahedralization*. Recently, Below et al. [2] proved that finding a minimal tetrahedralization of a convex polyhedron is NP-complete. Thus, it is significant to study which kind of polyhedra can be minimal tetrahedralized in polynomial time.

In this paper, we present some properties of the graph of polyhedra. Then we identify a class of polyhedra, called BP-polyhedron, and show that this kind of polyhedra can be minimally tetrahedralized in $O(n^2)$ time.

This paper is organized as follows. In Section 2, we present some properties of the graph of polyhedra. In Section 3, we prove the structure of the minimal tetrahedralization of BP-polyhedron. In Section 4, we present a quadratic time algorithm for identifying and tetrahedralizing such polyhedra. Finally, we give discussions in Section 5.

## 2 Properties of the graph of polyhedra

### 2.1 Definitions and assumptions

An *ordinary polyhedron* is a finite polyhedral complex embedded in $R^3$ and is topologically equivalent to a sphere. Therefore, it does not meet itself in a handle, or touch itself at a point or an edge, and the boundary of such a polyhedron forms a connected planar graph.

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A *diagonal* of a polyhedron is a line segment between two vertices that lies inside the polyhedron and does not intersect the polyhedron's boundary except at its endpoints.

Let $\mathcal{P}$ be an ordinary polyhedron. For simplicity, we assume that no four vertices of $\mathcal{P}$ are coplanar. This assumption implies that each facet of $\mathcal{P}$ is a triangle and that no diagonals intersect. A minimal tetrahedralization of $\mathcal{P}$ is a tetrahedralization that contains a minimal number of tetrahedra among all tetrahedralizations of $\mathcal{P}$. Throughout this paper, we will study the graph and the minimal tetrahedralization of $\mathcal{P}$. The graph of $\mathcal{P}$ , denoted by $G(\mathcal{P})$, is the graph with vertex set $V(\mathcal{P})$ and edge set $E(\mathcal{P})$, i.e., $G(\mathcal{P}) = (V(\mathcal{P}), E(\mathcal{P}))$.

Let $v_0v_1\cdots v_m$ be a path on $G(\mathcal{P})$. If $m > 2$, and there exist two vertices $p, q \in V(\mathcal{P})$, such that all $v_i, i = 0, 1, \cdots, m$, are adjacent to $p$ and $q$, then the subgraph induced from $p, q, v_i, i = 0, 1, \cdots, m$ is called a *bipyramid net* (for short, *BP-net*), denoted by $B(pv_0v_1\cdots v_mq)$. Moreover, if $v_0 = v_m$, that is, if $v_0v_1\cdots v_m$ is a cycle, then $B(pv_0v_1\cdots v_mq)$ is called a *bipyramid*. Vertices $p$ and $q$ are called poles of the BP-net or bipyramid. The edge $pq$ is the polar edge. Obviously, a BP-net can be obtained from a bipyramid by splitting $v_i$ of the path $pv_iq$ into two paths $pv_iq$ and $pv_i'q$, illustrated in Figure 1.

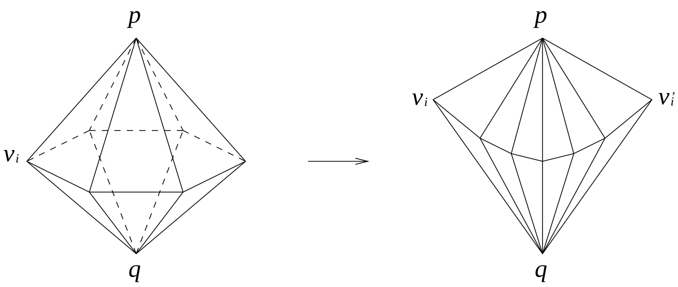


Figure 1: The relationship between bipyramid and BP-net

## 2.2 Properties of $G(\mathcal{P})$

For each edge $ab$ of $G(\mathcal{P})$, there must exist two vertices $c$ and $d$ such that $abc$ and $abd$ are two triangles in $G(\mathcal{P})$. Let $tri(ab)$ denote the set of two vertices that determine the two triangles sharing $ab$. The set of all adjacent vertices (i.e., neighbours) of a vertex $v$ in $G(\mathcal{P})$ is denoted by $N(v)$. The degree of $v$ is denoted by $deg(v)$. The cardinality of a set $S$ is denoted by $|S|$.

**Theorem 1**. If $|V(\mathcal{P})| > 4$, no two vertices of degree 3 are adjacent.

**Proof**. Refer to Figure 2. Suppose $ab \in E(\mathcal{P})$, and $deg(a) = deg(b) = 3$, Let $tri(ab) = \{c, d\}$. Since $deg(a) = 3$, $N(a) = \{b, c, d\}$; hence $tri(ac) = \{b, d\}$. Similarly, $tri(bc) = \{a, d\}$, $tri(ad) = \{b, c\}$, $tri(bd) = \{a, c\}$, and $tri(cd) = \{a, b\}$. Thus, $\mathcal{P} = abcd$ is a tetrahedron and $|V(\mathcal{P})| = 4$. This is a contradiction. Therefore no two vertices of degree 3 are adjacent. ♣

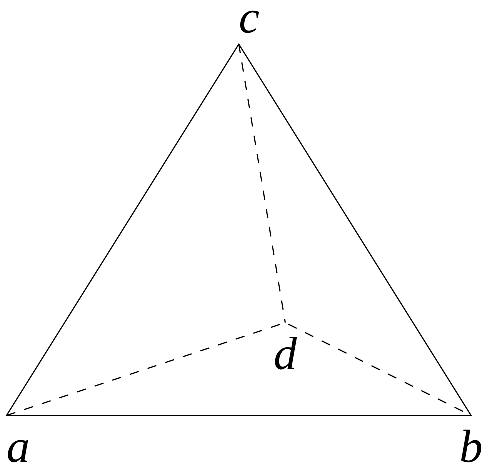


Figure 2: $a$ and $b$ are adjacent and their degrees are three

**Corollary 1**. Let $a, b \in V(\mathcal{P})$, and $deg(a) = deg(b) = 3$. Then $\mathcal{P}$ is a tetrahedron if and only if $a$ and $b$ are adjacent.

**Theorem 2**. Let $deg(a) = 3$, $a \in V(\mathcal{P})$. If $|V(\mathcal{P})| > 5$, then there are at most one vertex in $N(a)$ whose degree is less than 5.

**Proof**. Refer to Figure 3. Let $N(a) = \{a_1, a_2, a_3\}$. It follows from Theorem 1 that $deg(a_i) > 3, i = 1, 2, 3$. Suppose there exists one vertex $a_1$ whose degree is 4, and $N(a_1) = \{a, a_2, a_3, b\}$. Then $a_2b, a_3b \in E(\mathcal{P})$. Now consider the edge $a_2a_3$. If $tri(a_2a_3) = \{a, b\}$, then $\mathcal{P} = aa_1a_2a_3b$, and $|V(\mathcal{P})| = 5$, which contradicts the condition of the theorem. So $tri(a_2a_3) = \{a, c\}$. Thus, $deg(a_i) \geq 5, i = 2, 3$. This completes the proof. ♣

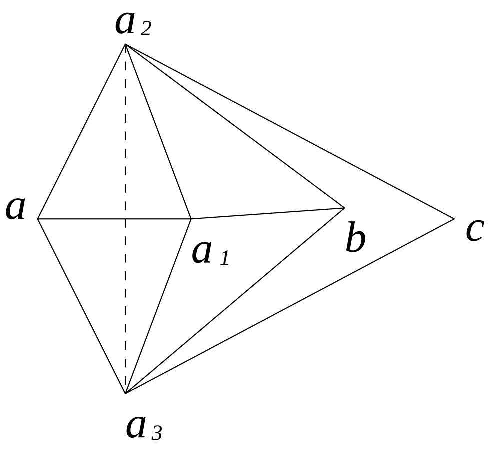


Figure 3: $deg(a) = 3, a_1 \in N(a)$ and $deg(a_1) = 4$

**Corollary 2**. Let $a, a_1, a_2 \in V(\mathcal{P})$, $deg(a) = 3$, and $deg(a_1) = deg(a_2) = 4$. Then $\mathcal{P}$ is a hexahedron if and only if $a_i \in N(a)$, $i = 1, 2$.

Moreover, for the vertex of degree 4 in $\mathcal{P}$, its neighbours have the following property.

**Theorem 3**. Let $deg(a) = 4, a \in V(\mathcal{P})$. If $|V(\mathcal{P})| > 6$, then there are at most two nonadjacent vertices in $N(a)$ whose degrees are less than 5.

**Proof**. Let $N(a) = \{a_1, a_2, a_3, a_4\}$. Consider the following two cases.

1. $deg(a_1) = 3$. Then $a_2a_4 \in E(\mathcal{P})$. So $deg(a_i) \geq 4, i = 2, 4$. If $deg(a_2) = 4$ or $deg(a_4) = 4$, then $\mathcal{P} = aa_1a_2a_3a_4$, $|V(\mathcal{P})| = 5$, which contradicts the condition of the theorem. Thus, $deg(a_i) > 4, i = 2, 4$. If $deg(a_3) = 3$, then $\mathcal{P} = aa_1a_2a_3a_4$, $|V(\mathcal{P})| = 5$, which contradicts the condition. The degree of $a_3$ can be equal to 4 (Refer to Figure 4).

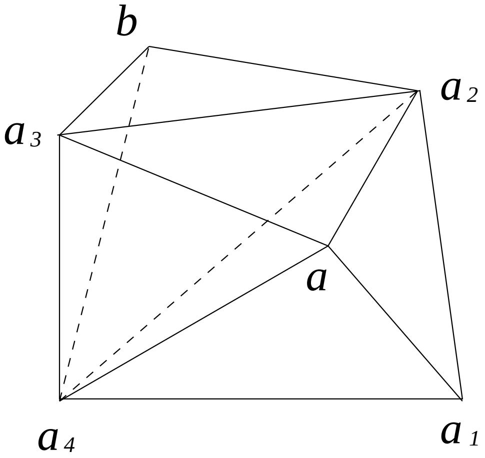


Figure 4: An octahedron (type I)

2. $deg(a_1) = 4$. Since $|V(\mathcal{P})| > 6$, $a_3 \notin N(a_1)$. Let $N(a_1) = \{a, a_2, a_4, b\}$, then $a_2b, a_4b \in E(\mathcal{P})$. If $deg(a_4) = 4$, then $N(a_4) = \{a, a_1, a_3, b\}$, and $tri(a_4b) = \{a_1, a_3\}$. So, $\mathcal{P} = aa_1a_2a_3a_4b$, which contradicts the condition $|V(\mathcal{P})| > 6$. Thus, $deg(a_4) > 4$. Similarly, we can prove that $deg(a_2) > 4$. The degree of $a_3$ can be equal to 4 (Refer to Figure 5). This ends the proof. ♣

From the definition of BP-net, we know that for a BP-net $B(pv_0v_1 \cdots v_mq)$, $deg(v_i) = 4, i = 1, \cdots, m-1$. Now we consider the inverse case.

**Theorem 4**. Let $a_1a_2 \cdots a_k$ be a path in $G(\mathcal{P}), k > 1$. If $deg(a_i) = 4, i = 1, \cdots, k$, then the subgraph induced from $a_i, i = 1, \cdots, k$ and their neighbours is a bipyramid or a BP-net.

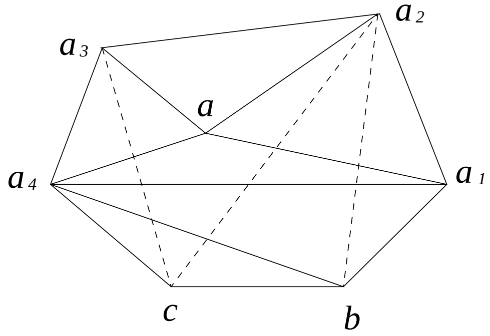


Figure 5: $deg(a) = 4$, $a_1, a_3 \in N(a)$, $deg(a_1) = deg(a_3) = 4$

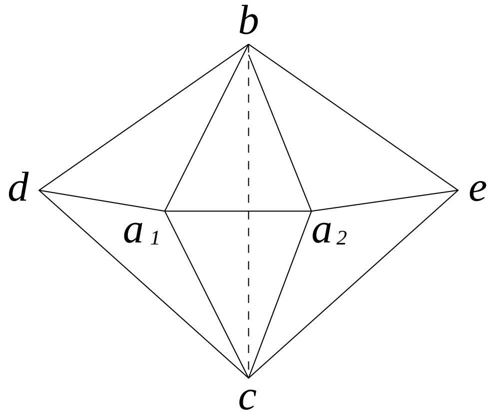


Figure 6: A BP-net

**Proof**. Consider the following four cases of $k$.

1. $k = 2$. Let $tri(a_1a_2) = \{b, c\}$, $N(a_1) = \{a_2, b, c, d\}$ and $N(a_2) = \{a_1, b, c, e\}$. Then $bd, cd, be, ce \in E(\mathcal{P})$. So the graph induced from $\{a_1, a_2, b, c, d, e\}$ will be a BP-net if $d \neq e$ or a bipyramid if $d = e$. Refer to Figure 6.

2. $k$=3. Consider the following two cases of $tri(a_1a_2)$.

   (a) $tri(a_1a_2) = \{b, c\}$. Then $N(a_2) = \{a_1, a_3, b, c\}$. So $a_3b, a_3c \in E(\mathcal{P})$. Let $d \in N(a_1)$, $e \in N(a_3)$, then $bd, cd, be, ce \in E(\mathcal{P})$. So the graph induced from $\{a_1, a_2, a_3, b, c, d, e\}$ will be a BP-net if $d \neq e$ or a bipyramid if $d = e$.

   (b) $tri(a_1a_2) = \{a_3, b\}$. Let $N(a_2) = \{a_1, a_3, b, c\}$. Hence $a_3c, bc \in E(\mathcal{P})$. If $c \in N(a_1)$, then $\mathcal{P} = a_1a_2a_3bc$ and $deg(a_3) = 3$, which is a contradiction. Thus, $N(a_1) = \{a_2, a_3, b, d\}$. Therefore $a_3d, bd \in E(\mathcal{P})$. Since $N(a_3) = \{a_1, a_2, c, d\}$, $cd \in E(\mathcal{P})$. Thus, the graph induced from $\{a_1, a_2, a_3, b, c, d\}$ is an octahedron (bipyramid), illustrated in Figure 7.

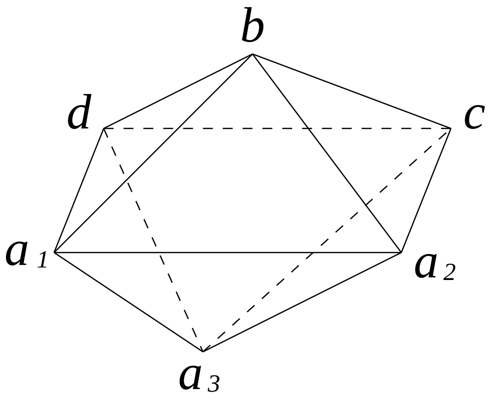


Figure 7: An octahedron (type II)

3. $4 \le k \le 6$. Consider the following two cases of $tri(a_1a_2)$.

   (a) If $tri(a_1a_2) = \{b, c\}$, then $N(a_i) = \{a_{i-1}, a_{i+1}, b, c\}$, $2 \le i \le k-1$. Let $d \in N(a_1)$, $e \in N(a_k)$. Then $bd, cd, be, ce \in E(\mathcal{P})$. Thus the graph induced from $\{a_1, \cdots, a_k, b, c, d, e\}$ will be a BP-net if $d \ne e$ or a bipyramid if $d = e$.

   (b) $tri(a_1a_2) = \{a_i, b\}$, $3 \le i \le k$. Using the same argument as Case 2(b), we can prove that the graph induced from $a_1, \cdots, a_k$ and their neighbours is an octahedron (bipyramid). Referring to Figure 7, we can set , for example, $c = a_4, d = a_5, b = a_6$.

4. $k > 6$. In this case, the graph induced from $a_1, \cdots, a_k$ and their neighbours cannot be an octahedron. So we can prove that this graph is a BP-net or bipyramid using the same argument as Case 2(a). ♣

From Theorem 3 and 4, we can prove the following three properties of octahedra.

**Corollary 3**. There are only two different types of graphs for octahedra.

For the two types of graphs for octahedra, one is called type I octahedron, illustrated in Figure 4; and the other is called type II octahedron, illustrated in Figure 7.

**Corollary 4**. Let $a, a_1, a_2 \in V(\mathcal{P})$, $a_1a_2 \in E(\mathcal{P})$, $deg(a_1) = 3$, and $deg(a) = deg(a_2) = 4$. Then $\mathcal{P}$ is a type I octahedron if and only if $a_i \in N(a)$, $i = 1, 2$.

**Corollary 5**. Let $a, a_1, a_2 \in V(\mathcal{P})$, $a_1a_2 \in E(\mathcal{P})$, and $deg(a) = deg(a_1) = deg(a_2) = 4$. Then $\mathcal{P}$ is a type II octahedron if and only if $a_i \in N(a)$, $i = 1, 2$.

# 3 Minimal tetrahedralizations of BP-polyhedra

## 3.1 Definitions

For an ordinary polyhedron, the tetrahedralization is an edge set that partitions the polyhedron into a number of tetrahedra that meet only at shared faces. The tetrahedralization consists of all the edges of the polyhedron and some of its diagonals. If $\mathcal{P}$ has a tetrahedralization in which there are no diagonals, then $\mathcal{P}$ is called a *stacked polyhedron* [14, 8]. Obviously, if $\mathcal{P}$ is a stacked polyhedron, then $\mathcal{P}$ can be constructed from a tetrahedra followed by successive addition of tetrahedra on some of their triangle facets.

Let $B(pv_0v_1 \cdots v_mq)$ be a BP-net in $G(\mathcal{P})$, which satisfies the following two conditions: (a) $pq$ is a diagonal or $pq \in E(\mathcal{P})$; and (b) each $pqv_jv_{j+1}$, $j = 0, 1, \cdots, m-1$, is a tetrahedron which is included in the polyhedron $\mathcal{P}$. Then the BP-net $B(pv_0v_1 \cdots v_mq)$ is called *proper*; and the union of all the tetrahedra $pqv_jv_{j+1}$, $j = 0, 1, \cdots, m-1$, is called *tetrahedralized BP-net*. It is clear that the tetrahedralized BP-net is a stacked polyhedron. For two proper BP-nets in $G(\mathcal{P})$, if the intersection of the interiors of their tetrahedralized BP-nets is empty, then they are *disjoint*.

Let $B(p^{(i)}v_0^{(i)}v_1^{(i)} \cdots v_{m_i}^{(i)}q^{(i)}), i = 1, \cdots, k$ be $k$ disjoint proper BP-nets in $G(\mathcal{P})$. After removing the vertices $v_j^{(i)}, j = 1, \cdots, m_i - 1, i = 1, \cdots, k$, and their adjacent edges from $G(\mathcal{P})$, and adding edges $p^{(i)}q^{(i)}, i = 1, \cdots, k$, we obtain a new polyhedron $P_1$. If $P_1$ is a stacked polyhedron, then $\mathcal{P}$ is called a *BP-polyhedron*. Furthermore, if $P_1$ is a BP-polyhedron, then $\mathcal{P}$ is called a *two-level* BP-polyhedron, and $P_1$ is called the generated BP-polyhedron of $\mathcal{P}$. Let $\mathcal{P}'$ be a (two-level) BP-polyhedron. If $\mathcal{P}$ is constructed from $\mathcal{P}'$ by successive addition of tetrahedra on the triangle facets, then $\mathcal{P}$ is called an *augmented* (two-level) BP-polyhedron.

Consider the process of removing vertices and adding edges (i.e., trim off $k$ disjoint proper BP-nets in $G(\mathcal{P})$) in the above definition. It is obvious that the order in which the process is done does not affect the results of Theorem 5 and 6 below because the BP-nets in Theorem 5 and 6 are disjoint.

Let $\Gamma$ be a tetrahedralization of $\mathcal{P}$. Let $\Lambda$ be a simply connected region on the surface of $\mathcal{P}$, which consists of the triangle facets. That is, the boundary of $\Lambda$ is a simple polygon on the surface of $\mathcal{P}$. Let $ab \in \Gamma$ and $ab$ be a diagonal of $\mathcal{P}$. If $a, b \in V(\Lambda)$, then $ab$ is called *type-I edge* on $\Lambda$. We also say that $\Lambda$ connects with type-I edge of $\Gamma$. If $a$ is an interior vertex of $\Lambda$, and $b \notin V(\Lambda)$, then ab is called *type-II edge* on $\Lambda$. We also say that $\Lambda$ connects with type-II edge of $\Gamma$. Let $tetra(abc)$ denote the set of vertices that form the tetrahedra with $abc$ in $\Gamma$. Obviously, $|tetra(abc)| = 1$ or $2$. Each vertex in

$tetra(abc)$ is called the *tetra-vertex* of $abc$.

### 3.2 BP-polyhedra

**Theorem 5.** Let $\mathcal{P}$ be a BP-polyhedron, and $p^{(i)}q^{(i)}, i = 1, \cdots, k$ be all the polar edges of the disjoint proper BP-nets in $G(\mathcal{P})$. Then $\{p^{(i)}q^{(i)} : i = 1, \cdots, k\} \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$.

**Proof.** Let $B(p^{(i)}v_0^{(i)}v_1^{(i)} \cdots v_{m_i}^{(i)}q^{(i)}), i = 1, \cdots, k$ be $k$ disjoint proper BP-nets in $G(\mathcal{P})$. After removing the vertices $v_j^{(i)}, j = 1, \cdots, m_i - 1, i = 1, \cdots, k$, and their adjacent edges from $G(\mathcal{P})$, and adding edges $p^{(i)}q^{(i)}, i = 1, \cdots, k$, we obtain a new polyhedron $Q$. From the definition of the BP-polyhedron we know that $Q$ is a stacked polyhedron. Since each $B(p^{(i)}v_0^{(i)}v_1^{(i)} \cdots v_{m_i}^{(i)}q^{(i)})$ and the polar edge $p^{(i)}q^{(i)}$ also construct a stacked polyhedron, the edge set $\{p^{(i)}q^{(i)} : i = 1, \cdots, k\} \cup E(\mathcal{P})$ is a tetrahedralization of $\mathcal{P}$. $\{p^{(i)}q^{(i)} : i = 1, \cdots, k\}$ may contain edges that belong to $E(\mathcal{P})$ or edges that are coincident. Without loss of generality, we can assume that all the edges $p^{(i)}q^{(i)}, i = 1, \cdots, k$ are different diagonals. Now we will show that $\{p^{(i)}q^{(i)} : i = 1, \cdots, k\} \cup E(\mathcal{P})$ is a minimal tetrahedralization by contradiction.

Suppose that $\mathcal{T}$ is a minimal tetrahedralization of $\mathcal{P}$ and $|\mathcal{T}| < k + |E(\mathcal{P})|$. Let $T = \mathcal{T} - E(\mathcal{P})$, $T_0 = T \cap \{p^{(i)}q^{(i)} : i = 1, \cdots, k\}$, $T_1 = T - T_0$. Since the BP-net is not a tetrahedralization, there must exist some edges of $T$ whose endpoints belong to the BP-net. Thus, we can classify the BP-nets into two sets: $\beta_0$ and $\beta_1$. For each BP-net in $\beta_0$, its polar edge belongs to $T_0$, and for each BP-net in $\beta_1$, its polar edge does not belong to $T$. First, we prove the following lemma.

**Lemma 1.** Each BP-net $B(pv_0v_1 \cdots v_mq)$ in $\beta_1$ must connect with at least
(1) two type-I edges of $T$, or
(2) one type-I edge and one type-II edge of $T$, or
(3) three type-II edges of $T$.

**Proof.** Each triangle in $G(\mathcal{P})$ has only one tetra-vertex and each triangle in the interior of $\mathcal{P}$ has exactly two tetra-vertices. From the definition of BP-net, we know that $m \geq 3$. Consider the triangle $v_0pv_1$. Its tetra-vertex cannot be $q$ since $B(pv_0v_1 \cdots v_mq)$ is in $\beta_1$; that is, $pq \notin T$. We have the following three cases.

1. $v_2 \in tetra(v_0pv_1)$. Then $v_0v_2 \in T$, which is a type-I edge. Consider the interior triangle $v_0v_2p$. $v_1$ is one tetra-vertex. The other tetra-vertex may be $v_i (3 \leq i \leq m)$, or the vertex $a$ which does not belong to $B(pv_0v_1 \cdots v_mq)$. So $v_0v_i \in T$ or $v_2a \in T$. In total, the BP-net connects with at least two type-I edges, or one type-I and one type-II edge.
2. $v_i \in tetra(v_0pv_1), 3 \leq i \leq m$. Then $v_0v_i, v_1v_i \in T$, which are type-I edges. Thus the BP-net connects with at least two type-I edges.
3. $b \in tetra(v_0pv_1)$ and $b \notin B(pv_0 \cdots v_mq)$. Then $v_1b \in T$, which is a type-II edge. There are three cases for the tetra-vertex of triangle $v_1pv_2$.
   (a) $v_i \in tetra(v_1pv_2), 3 \leq i \leq m$. Then $v_1v_i \in T$, which is a type-I edge. Thus, the BP-net connects with at least one type-I edge and one type-II edge.
   (b) $c \in tetra(v_1pv_2)$ and $c \notin B(pv_0 \cdots v_mq)$. Then $v_1c, v_2c \in T$, which are type-II edges. Thus, the BP-net connects with at least three type-II edges.
   (c) $b \in tetra(v_1pv_2)$. Then $v_2b \in T$. Consider the triangle $v_2pv_3$. If $m > 3$, the tetra-vertex of triangle $v_2pv_3$ may be $v_i (4 \leq i \leq m)$, or the vertex $b$, or the vertex $b' \notin B(pv_0 \cdots v_mq)$. Thus $v_2v_i \in T_0$, or $v_3b \in T$, or $v_2b', v_3b' \in T$. Therefore, $B(pv_0 \cdots v_mq)$ connects with at least one type-I edge and two type-II edges, or at least three type-II edges. If $m$=3, we have the following two cases:
      i. $d \in tetra(v_2pv_3)$ and $d \notin B(pv_0 \cdots v_mq)$ . Then $v_2d \in T$, which is a type-II edge. Thus, $B(pv_0 \cdots v_mq)$ connects with at least three type-II edges.
      ii. $b \in tetra(v_2pv_3)$. Then $v_3b \in T$. We can discuss the symmetric triangles $v_0qv_1$, $v_1qv_2$, $v_2qv_3$ using the above argument. Either we prove the lemma, or we get the same case as above; that is, $qb \in T$. Let $C$ be the convex hull of the vertices $\{v_0, v_1, v_2, v_3, p, q, b\}$. Obviously, $C \cup \{v_1b, v_2b\}$ is a tetrahedralization of $C$, and it is a subgraph of $T$. Since $T$ is a minimal tetrahedralization of $\mathcal{P}$, $\{v_1b, v_2b\}$ is a minimal tetrahedralization of $C$. However, this is a contradiction because $\{pq\}$ is a minimal tetrahedralization of $C$. This completes the proof of Lemma 1. ♣

Each type-I edge connects with only one BP-net, and each type-II edge connects with at most two BP-nets. It follows from Lemma 1 that the total number of edges in $T_1$ that connect with the BP-nets in $\beta_1$ is greater than $|\beta_1|$. Since we supposed that all the edges $p^{(i)}q^{(i)}, 1 \leq i \leq k$ are different, the total number of edges in $T_0$ is $|\beta_0|$. Thus

$$|T| = |T_1| + |T_0| > |\beta_1| + |\beta_0| = k.$$

This contradicts the assumption that $T \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$. Therefore, $\{p^{(i)}q^{(i)} : i = 1, \cdots, k\} \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$. This ends the proof of Theorem 5. ♣

Using a similar method to that used in the above proof, we can prove the minimal tetrahedralizations of augmented BP-polyhedra.

**Corollary 6**. Let $\mathcal{P}$ be an ordinary polyhedron. After removing all the degree 3 vertices in $G(\mathcal{P})$ recursively, we obtain a new polyhedron $\mathcal{P}'$. If $\mathcal{P}'$ is a BP-polyhedron, then $\{p^{(i)}q^{(i)} : i = 1, \cdots, k\} \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$, where $p^{(i)}q^{(i)}$, $i = 1, \cdots, k$ are the polar edges of the $k$ disjoint proper BP-nets on $G(\mathcal{P}')$.

### 3.3 Two-level BP-polyhedra

**Lemma 2**. Let $B = B(xz_0z_1 \cdots z_my)$ be a proper BP-net in $G(\mathcal{P})$. By deleting the interior edges of $B$ and adding edge $xy$, we obtain a new proper BP-net, denoted by $\tilde{B}_1 = B(pv_0v_1v_2v_3q)$, where $v_1 = x, v_2 = y$, $p = z_0, q = z_m$. Let $T$ be a minimal tetrahedralization of $\mathcal{P}$, and $\overline{B}_1$ be the graph induced from $p, q, v_0, v_1, v_2, v_3, z_i (0 \leq i \leq m)$ and bounded by $pv_0, pv_3, qv_0, qv_3$. If $xy, pq \notin T$, then $T$ must have at least
(1) three type-I edges on $\overline{B}_1$, or
(2) two type-I and one type-II edges on $\overline{B}_1$, or
(3) one type-I and three type-II edges on $\overline{B}_1$, or
(4) five type-II edges on $\overline{B}_1$.

**Proof**. We will only prove the lemma for $m = 3$. It can be similarly proved for $m > 3$. Refer to Figure 8.

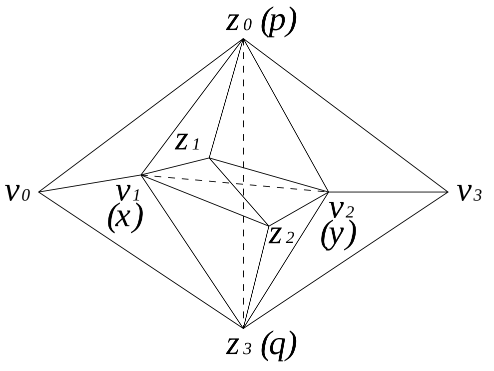


Figure 8: $v_1 = x, v_2 = y, p = z_0, q = z_3$

Consider the triangle $z_0v_1z_1$. $v_2, z_3 \notin tetra(z_0v_1z_1)$ since $v_1v_2, z_0z_3 \notin T$. Thus, $tetra(z_0v_1z_1)$ may contain $z_2$, or $v_0$, or $v_3$, or the vertex $c$ which does not belong to $\overline{B}_1$. Therefore, we have the following cases.

1. $z_2 \in tetra(z_0v_1z_1)$. Then $z_0z_2 \in T$, which is a type-I edge. Consider the interior triangle $z_0v_1z_2$. $z_1 \in tetra(z_0v_1z_2)$, and the other vertex in $tetra(z_0v_1z_2)$ may be $v_0$, or $v_3$, or the vertex $a$ which does not belong to $\overline{B}_1$.
    (a) $v_0 \in tetra(z_0v_1z_2)$, then $z_2v_0 \in T$, which is a type-I edge. Consider triangle $z_0z_2v_0$.
        i. If $v_2$ or $v_3$ belongs to $tetra(z_0z_2v_0)$, then $v_0v_2$ or $v_0v_3 \in T$, which is type-I edge. In total, there are at least three type-I edges on $\overline{B}_1$.
        ii. If $a' \in tetra(z_0z_2v_0)$ which does not belong to $\overline{B}_1$, then $z_2a' \in T$, which is type-II edge. In total, there are at least two type-I edges and one type-II edge on $\overline{B}_1$.
    (b) $v_3 \in tetra(z_0v_1z_2)$, then $v_1v_3, z_2v_3 \in T$, which are type-I edges. Thus, there are at least three type-I edges on $\overline{B}_1$.
    (c) $a \in tetra(z_0v_1z_2)$, then $v_1a, z_2a \in T$, which are type-II edges. Considering triangle $z_0z_2v_2$, at least one type-I or type-II edge is incident on $z_0$, or $z_2$, or $v_2$. In total, there are at least two type-I edges and two type-II edges, or one type-I edge and three type-II edges on $\overline{B}_1$.
2. $v_0 \in tetra(z_0v_1z_1)$. Then $z_1v_0 \in T$, which is a type-I edge. Consider the interior triangle $z_1v_0v_1$. $z_0 \in tetra(z_1v_0v_1)$, and the other vertex in $tetra(z_1v_0v_1)$ may be $z_2$, or $z_3$, or $v_3$, or the vertex $b$ which does not belong to $\overline{B}_1$.
    (a) $z_2 \in tetra(z_1v_0v_1)$, then $z_2v_0 \in T$, which is a type-I edge. Consider triangle $z_1z_2v_0$.
        i. If $z_0$, or $z_3$, or $v_2$, or $v_3$ belongs to $tetra(z_1z_2v_0)$, then at least one type-I edge is incident on $z_1$, or $z_2$, or $v_0$. In total, there are at least three type-I edges on $\overline{B}_1$.
        ii. If $b' \in tetra(z_1z_2v_0)$ which does not belong to $\overline{B}_1$, then $z_1b', z_2b' \in T$, which are type-II edges. In total, there are at least two type-I edges and two type-II edges on $\overline{B}_1$.
    (b) $z_3 \in tetra(z_1v_0v_1)$, then $z_1z_3 \in T$, which is type-I edge. Considering triangle $z_1z_3v_0$, at least one type-I or one type-II edge is incident on $z_1$, or $z_3$, or $v_0$. In total, there are at least three type-I edges, or two type-I edges and one type-II edge on $\overline{B}_1$.
    (c) $v_3 \in tetra(z_1v_0v_1)$, then $z_1v_3, v_0v_3, v_1v_3 \in T$, which are type-I edges. In total, there are at least four type-I edges on $\overline{B}_1$.

(d) $b \in tetra(z_1v_0v_1)$, then $z_1b, v_1b \in T$, which are type-II edges. Considering triangle $z_1z_2v_2$, at least one type-I or type-II edge is incident on $z_1$, or $z_2$, or $v_2$. In total, there are at least two type-I edges and two type-II edges, or one type-I edge and three type-II edges on $\overline{B}_1$.

3. $v_3 \in tetra(z_0v_1z_1)$. Then $z_1v_3, v_1v_3 \in T$, which are type-I edges. Considering the interior triangle $z_1v_1v_3$, at least one type-I or type-II edge is incident on $z_1$, or $v_1$, or $v_3$. In total, there are at least three type-I edges, or two type-I edges and one type-II edge on $\overline{B}_1$.

4. $c \in tetra(z_0v_1z_1)$ and $c \notin \overline{B}_1$. Then $z_1c, v_1c \in T$, which are type-II edges. Consider triangle $z_1v_1z_2$. If $z_0 \in tetra(z_1v_1z_2)$, then the intersection of the interiors of tetrahedra $z_0z_1v_1z_2$ and $z_0v_1z_1c$ is not empty. Thus $z_0 \notin tetra(z_1v_1z_2)$.

   (a) $z_3 \in tetra(z_1v_1z_2)$, then $z_1z_3 \in T$, which is a type-I edge. Considering triangle $z_1z_2v_2$, at least one type-I or type-II edge is incident on $z_1$, or $z_2$, or $v_2$. In total, there are at least two type-I edges and two type-II edges, or one type-I edge and three type-II edges on $\overline{B}_1$.

   (b) $v_3 \in tetra(z_1v_1z_2)$, then $v_1v_3, z_1v_3, z_2v_3 \in T$, which are type-I edges. In total, there are three type-I edges and two type-II edges on $\overline{B}_1$.

   (c) $v_0 \in tetra(z_1v_1z_2)$, then $z_1v_0, z_2v_0 \in T$, which are type-I edges. In total, there are two type-I edges and two type-II edges on $\overline{B}_1$.

   (d) $c' \in tetra(z_1v_1z_2)$, then $v_1c', z_1c', z_2c' \in T$, which are type-II edges. In total, there are at least five type-II edges on $\overline{B}_1$.

   (e) $c \in tetra(z_1v_1z_2)$, then $z_2c \in T$, which is type-II edge. Consider triangle $z_1z_2v_2$.

      i. $z_0, z_3, v_0$ or $v_3$ belongs to $tetra(z_1z_2v_2)$, at least one type-I edge is incident on $z_1$, or $z_2$, or $v_2$. In total, there are at least one type-I edge and three type-II edges on $\overline{B}_1$.

      ii. $c'' \in tetra(z_1z_2v_2)$ and $c'' \notin \overline{B}_1$, then $z_1c'', z_2c'', v_2c'' \in T$. In total, there are at least six type-II edges on $\overline{B}_1$.

      iii. $c \in tetra(z_1v_2z_2)$, then $v_2c \in T$. Consider the other triangles in $\overline{B}_1$ (except triangle $z_0v_1z_1, z_1v_1z_2, z_1v_2z_2$). If there is at least one triangle $xyz$ whose tetra-vertex is $d \neq c$, then $\overline{B}_1$ connects with at least one type-I edge and four type-II edges, or five type-II edges; otherwise, all the vertices in $\overline{B}_1$ are adjacent to $c$. Let $C$ be the convex hull of the vertex set $V(\overline{B}_1) \cup \{c\}$. Obviously, $C \cup \{z_1c, z_2c, v_1c, v_2c\}$ is a tetrahedralization of $C$, and it is a subgraph of $T$. Since $T$ is a minimal tetrahedralization of $\mathcal{P}$, $C \cup \{z_1c, z_2c, v_1c, v_2c\}$ is a minimal tetrahedralization of $C$. However, since no four vertices of $\mathcal{P}$ are coplanar (this is the assumption in Section 2.1), $v_1v_2$,and $z_0z_3$ cannot intersect. Thus, $\{v_1v_2, z_0z_3\}$ is a minimal tetrahedralization of $C$. This is a contradiction. Therefore, this case cannot occur. This completes the proof of Lemma 2. ♣

Using a similar method to that used in the above proof, we can prove the following three lemmas. We omit the proofs due to the length of this paper.

**Lemma 3**. Let $B = B(xz_0z_1 \cdots z_my)$ be a proper BP-net in $G(\mathcal{P})$. By deleting the interior edges of $B$ and adding edge $xy$, we obtain a new proper BP-net, denoted by $\tilde{B}_2 = B(pv_0v_1v_2v_3q)$, where $v_2 = x, v_3 = y$, $p = z_0, q = z_m$. Let $T$ be a minimal tetrahedralization of $\mathcal{P}$, and $\overline{B}_2$ be the graph induced from $p, q, v_0, v_1, v_2, v_3, z_i (0 \leq i \leq m)$ and bounded by $pv_0, pv_3, qv_0, qv_3$. If $xy, pq \notin T$, then $T$ must have at least
(1) three type-I edges on $\overline{B}_2$, or
(2) two type-I and one type-II edges on $\overline{B}_2$, or
(3) one type-I and three type-II edges on $\overline{B}_2$, or
(4) five type-II edges on $\overline{B}_2$.

**Lemma 4**. Let $B = B(xz_0z_1 \cdots z_my)$ be a proper BP-net in $G(\mathcal{P})$. By deleting the interior edges of $B$ and adding edge $xy$, we obtain a new proper BP-net, denoted by $\tilde{B}_3 = B(pv_0v_1v_2v_3q)$, where $p = x, v_2 = y$, $v_1 = z_0, v_3 = z_m$. Let $T$ be a minimal tetrahedralization of $\mathcal{P}$, and $\overline{B}_3$ be the graph induced from $p, q, v_0, v_1, v_2, v_3, z_i (0 \leq i \leq m)$ and bounded by $pv_0, pv_3, qv_0, qv_3$. If $xy, pq \notin T$, then $T$ must have at least
(1) three type-I edges on $\overline{B}_3$, or
(2) two type-I and one type-II edges on $\overline{B}_3$, or
(3) one type-I and three type-II edges on $\overline{B}_3$, or
(4) five type-II edges on $\overline{B}_3$.

**Lemma 5**. Let $B = B(xz_0z_1 \cdots z_my)$ be a proper BP-net in $G(\mathcal{P})$. By deleting the interior edges of $B$ and adding edge $xy$, we obtain a new proper BP-net, denoted by $\tilde{B}_4 = B(pv_0v_1v_2v_3q)$, where $p = x, v_3 = y$, $v_2 = z_0$. Let $T$ be a minimal tetrahedralization of $\mathcal{P}$, and $\overline{B}_4$ be the graph induced from $p, q, v_0, v_1, v_2, v_3, z_i (0 \leq i \leq m)$ and bounded by $pv_0, pz_m, qv_0, qv_3, z_mv_3$. If $xy, pq \notin T$, then $T$ must have at least
(1) three type-I edges on $\overline{B}_4$, or

(2) two type-I and one type-II edges on $\overline{B}_4$, or
(3) one type-I and three type-II edges on $\overline{B}_4$, or
(4) five type-II edges on $\overline{B}_4$.

From Lemma 2-5, we can prove the following property.

**Lemma 6**. Let $B = B(xz_0z_1 \cdots z_m y)$ be a proper BP-net in $G(\mathcal{P})$. By deleting the interior edges of $B$ and adding edge $xy$, we obtain a new proper BP-net, denoted by $\tilde{B} = B(pv_0v_1 \cdots v_l q)$, where $xy$ is an edge of $\tilde{B}$. Let $T$ be a minimal tetrahedralization of $\mathcal{P}$, and $\overline{B}$ be the graph induced from the vertices of $B$ and $\tilde{B}$. If $xy, pq \notin T$, then $T$ must have at least
(1) three type-I edges on $\overline{B}$, or
(2) two type-I and one type-II edges on $\overline{B}$, or
(3) one type-I and three type-II edges on $\overline{B}$, or
(4) five type-II edges on $\overline{B}$.

**Proof**. From the structure of $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$, and $\tilde{B}_4$ in Lemma 2,3,4 and 5, we know that $\tilde{B}$ must contain at least one of $\tilde{B}_i, i = 1, 2, 3, 4$ as its subgraph. Thus, Lemma 6 is implied by Lemma 2-5. ♣

**Theorem 6**. Let $\mathcal{P}$ be a two-level BP-polyhedron, and $P_1$ be the generated BP-polyhedron of $\mathcal{P}$. Let $PE(\mathcal{P})$ be the set of all the polar edges of the disjoint proper BP-nets in $G(\mathcal{P})$, and $PE(P_1)$ be the set of all the polar edges of the disjoint proper BP-nets in $G(P_1)$. If each BP-net whose polar edge belongs to $PE(P_1)$ only contains one edge in $PE(\mathcal{P})$, then $PE(\mathcal{P}) \cup PE(P_1) \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$.

**Proof**. Let $B_i = B(x^{(i)} z_0^{(i)} z_1^{(i)} \cdots z_{m_i}^{(i)} y^{(i)})$, $i = 1, \cdots k_1$ be $k_1$ disjoint proper BP-nets in $G(\mathcal{P})$. By deleting the interior edges of $B_i$ and adding edges $x^{(i)}y^{(i)}$, we obtain the generated BP-polyhedron $P_1$.

Let $\tilde{B}_j = B(p^{(j)} v_0^{(j)} v_1^{(j)} \cdots v_{l_j}^{(j)} q^{(j)})$, $j = 1, \cdots k_2$ be $k_2$ disjoint proper BP-nets in $G(P_1)$. Thus, $PE(\mathcal{P}) = \{x^{(i)}y^{(i)} : i = 1, \cdots k_1\}$, $PE(P_1) = \{p^{(j)}q^{(j)} : j = 1, \cdots k_2\}$. It is obvious that $PE(\mathcal{P}) \cup PE(P_1) \cup E(\mathcal{P})$ is a tetrahedralization of $\mathcal{P}$. Without loss of generality, we can assume that all the edges in $PE(\mathcal{P})$ and $PE(P_1)$ are different diagonals. Now we will prove , by contradiction, that $PE(\mathcal{P}) \cup PE(P_1) \cup E(\mathcal{P})$ is a minimal tetrahedralization.

Suppose that $\mathcal{T}$ is a minimal tetrahedralization of $\mathcal{P}$ and $|\mathcal{T}| < k_1 + k_2 + |E(\mathcal{P})|$. Let $T = \mathcal{T} - |E(\mathcal{P})|$. From the condition of the theorem, we know that each $\tilde{B}_j$ only contains one $x^{(i)}y^{(i)}$. Let $T_0$ be a subset of $PE(\mathcal{P})$ with $k_2$ elements and each element of $T_0$ be an edge of $\tilde{B}_j$, $j = 1, \cdots k_2$.

Let $T_1 = PE(\mathcal{P}) - T_0$, $T_{00} = T \cap T_0$, $T_{01} = T_0 - T_{00}$. Let $\beta_{00}$ be the set of BP-nets whose polar edges belong to $T_{00}$, and $\beta_{01}$ be the set of BP-nets whose polar edges belong to $T_{01}$. Let $\tilde{\beta}_0$ be the set of BP-nets in $G(P_1)$ which contains the polar edges of $\beta_{00}$, and $\tilde{\beta}_1$ be the set of BP-nets in $G(P_1)$ which contains the polar edges of $\beta_{01}$. Let $\beta_1$ be the set of BP-nets whose polar edges belong to $T_1$. Consider the following two cases.

Case 1. For each BP-net in $\beta_1$ and $\tilde{\beta}_0$ whose polar edge does not belong to $T$, from Theorem 5 we know that the BP-net must connect with at least two type-I edges of $T$, or one type-I edge and one type-II edge of $T$, or three type-II edges of $T$.

Case 2. Let $\overline{B}$ be the graph induced from the vertices of $B'$ in $\beta_{01}$ and $\tilde{B}'$ in $\tilde{\beta}_1$ where $\tilde{B}'$ contains the polar edge of $B'$.

Case 2.1. If the polar edge of $\tilde{B}'$ belongs to $T$, and the polar edge of $B'$ does not belong to $T$, then it follows from Theorem 5 that $\overline{B}$(in fact, $B'$) must connect with at least two type-I edges of $T$, or one type-I edge and one type-II edge of $T$, or three type-II edges of $T$.

Case 2.2. If the polar edge of $\tilde{B}'$ does not belong to $T$, and the polar edge of $B'$ belongs to $T$, then it follows from Theorem 5 that $\overline{B}$(in fact, $\tilde{B}'$) must connect with at least two type-I edges of $T$, or one type-I edge and one type-II edge of $T$, or three type-II edges of $T$.

Case 2.3. If the polar edges of $\tilde{B}'$ and $B'$ do not belong to $T$, then it follows from Lemma 6 that $\overline{B}$ must connect with at least three type-I edges of $T$, or two type-I and one type-II edges of $T$, or one type-I and three type-II edges of $T$, or five type-II edges of $T$.

It follows from the above cases that

$$|T| > k_1 + k_2.$$

Thus, this contradicts that $T \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$. Therefore $PE(\mathcal{P}) \cup PE(P_1) \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$. ♣

Similarly, we can prove the minimal tetrahedralizations of augmented two-level BP-polyhedra.

**Corollary 7**. Let $\overline{\mathcal{P}}$ be an ordinary polyhedron. By removing all the degree 3 vertices in $G(\overline{\mathcal{P}})$ recursively, we obtain a new polyhedron $\mathcal{P}$. If $\mathcal{P}$ is the two-level BP-polyhedron described in Theorem 6, then $PE(\mathcal{P}) \cup PE(P_1) \cup E(\mathcal{P})$ is a minimal tetrahedralization of $\mathcal{P}$.

# 4 Algorithm

**Input**: An ordinary polyhedron $P$.
**Output**: $T$ (If $P$ is a BP-polyhedron, then $T$ is the tetrahedron set of the minimal tetrahedralization of $P$.)
**1**. Label the vertices of $V(P)$.
**2**. Represent $P$ in adjacency-list form. For each $u \in V(P)$, sort the vertex labels in its adjacency list $N(u)$.
**3**. $V \leftarrow V(P)$, $U \leftarrow \emptyset$ (empty set).
**4**. Arbitrarily select a vertex $u \in V$.
**4.1**. **If** $deg(u) = 3$ (suppose $N(u) = \{u_1, u_2, u_3\}$), **then**

**4.1(a)**. If there exists one $u_i (i = 1, 2, 3)$ of degree 3, then ($V = \{u, u_1, u_2, u_3\}$ from Corollary 1) $T \leftarrow \{uu_1u_2u_3\}$. Stop. T is an optimal tetrahedralization.
**4.1(b)**. If $deg(u_i) > 3, i = 1, 2, 3$, then $T \leftarrow \{uu_1u_2u_3\}$, $V \leftarrow V - \{u\}$. Update the adjacency list of $u_i, i = 1, 2, 3$. For $i = 1, 2, 3$, if $u_i \in U$, then $U \leftarrow U - \{u_i\}$, $V \leftarrow \{u_i\}$, go to Sept 4.
**4.2**. **If** $deg(u) > 4$, **then**
$U \leftarrow \{u\}$, $V \leftarrow V - \{u\}$. If $V = \emptyset$, then stop. P is not a BP-polyhedron. If $V \neq \emptyset$, then go to Sept 4.
**4.3**. **If** $deg(u) = 4$ (suppose $N(u) = \{u_1, u_2, u_3, u_4\}$), **then**
**4.3(a)**. If there exist at least one $u_i$ of degree 4 (suppose $deg(u_1) = 4$), then there exists a BP-net (Theorem 4). Suppose the BP-net is $B(pu'uu_1u''q)$. If $pq$ is a diagonal and all the edges in $B(pu'uu_1u''q)$ belong to $E(P)$, then $V \leftarrow V - \{u, u_1\}$, $T \leftarrow \{pqu'u, pquu_1, pqu_1u''\}$. If $|V \cup U| = \emptyset$, then stop. T is an optimal tetrahedralization. If $|V \cup U| \neq \emptyset$, then update the adjacency list of $p, q, u', u''$ (Note that $pq$ is a new edge), and move $p, q, u'$ or $u''$ from $U$ to $V$ if it is in $U$.
**4.3(b)**. If there exists one $u_i$ of degree 3 (suppose $deg(u_1) = 3$) (Theorem 2), then change $u$ into $u_1$, go to step 4.1.
**4.3(c)**. If $deg(u_i) > 4, i = 1, 2, 3, 4$, then $U \leftarrow \{u, u_1, u_2, u_3, u_4\}$, $V \leftarrow V - \{u, u_1, u_2, u_3, u_4\}$. If $V = \emptyset$, then stop. P is not a BP-polyhedron. If $V \neq \emptyset$, then go to Sept 4.

**Theorem 7**. The minimal tetrahedralization of a BP-polyhedron can be found in $O(n^2)$ time and $O(n)$ space.

**Proof**. Since $G(\mathcal{P})$ is a planar graph, $|E(\mathcal{P})| = O(n)$. Step 1, 3 and 4 take $O(n)$ time and $O(n)$ space. Sorting in Step 2 needs $O(n \log n)$ time and $O(n)$ space. Updating the adjacency list in Step 4.1(b) and 4.3(a) needs $O(\log n)$ time and $O(n)$ space because the vertices in $N(u)$ have been sorted. Checking if $u_i \in U$ in Step 4.1(b) and 4.3(a) needs $O(n)$ time and $O(n)$ space. Checking if $pq$ is a diagonal in Step 4.3(a) needs $O(n)$ time and $O(n)$ space. Therefore, computing the minimal tetrahedralization of a BP-polyhedron takes $O(n^2)$ time and $O(n)$ space. ♣

# 5 Discussions

From the definition of the BP-polyhedron, we know that there are two conditions, one relating to BP-net (i.e, the BP-nets are proper and disjoint); and the other relating to the stacked polyhedron (i.e, the new polyhedron constructed is a stacked polyhedron). For a polyhedron $P$ with BP-nets, if either of the two conditions cannot be satisfied, then $\{p^{(i)}q^{(i)} : i = 1, \cdots, k\} \cup E(P)$ may be not a minimal tetrahedralization of P, where $p^{(i)}q^{(i)}$ are the polar edges of the BP-nets on P. Here we give two counterexamples.

The first counterexample is the Schonhardt's polyhedron depicted in Figure 9. The way to construct this polyhedron is to begin with a triangular prism. Fix the bottom triangle so that it cannot move while the top triangle face is twisted by a small amount so that each rectangular face of the prism folds into two triangles with a reflex edge between them.

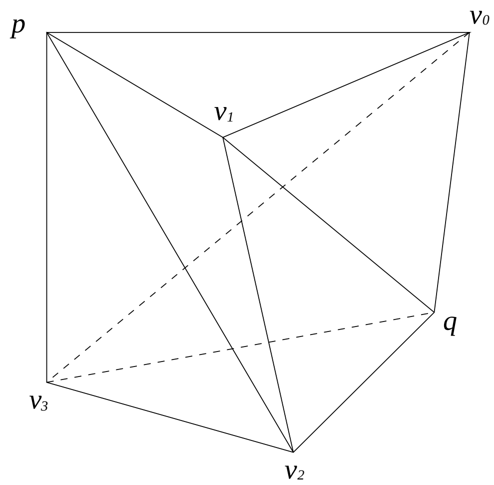


Figure 9: Schonhardt's polyhedron

The Schonhardt's polyhedron is also a BP-net, specifically, a bipyramid. As shown in Figure 9, $B(pv_0v_1v_2v_3q)$ is a BP-net. However, it is not a proper BP-net. Thus, we cannot obtain the result of Theorem 5. In fact, it is well-known that the Schonhardt's polyhedron cannot be tetrahedralized [16].

The second counterexample is a convex polyhedron constructed from the Schonhardt's polyhedron. Let $p_1p_2p_3q_1q_2q_3$ be a Schonhardt's polyhedron, where $p_1p_2p_3$ is the top triangle and $q_1q_2q_3$ is the bottom triangle. Let $B(p_1q_1a_1a_2p_2q_2)$, $B(p_2q_2b_1b_2p_3q_3)$ and $B(p_3q_3c_1c_2p_1q_1)$ be disjoint proper BP-nets adding to the Schonhardt's polyhedron such that the reflex edges $p_1q_2, p_2q_3, p_3q_1$ are the polar edges of the BP-nets respectively, illustrated in Figure 10. We can arrange $a_i, b_i, c_i, i = 1, 2$ such that the constructed polyhedron $P$ is convex.

After removing the vertices $a_i, b_i, c_i$ and their adjacent edges from $P$ and adding the polar edges $p_1q_2$, $p_2q_3$, $p_3q_1$, we get the Schonhardt's polyhedron which is not a stacked polyhedron. Thus, we cannot guarantee that $\{p_1q_2, p_2q_3, p_3q_1\} \cup E(P)$ is a minimal tetrahedralization of P.

In Section 3.1, we define the two-level BP-polyhedron. Similarly, we can define $h$-level BP-polyhedron recursively. Let $B(p^{(i)}v_0^{(i)}v_1^{(i)} \cdots v_{m_i}^{(i)}q^{(i)})$, $i = 1, \cdots, k$ be $k$ disjoint proper BP-nets in $G(\mathcal{P})$.

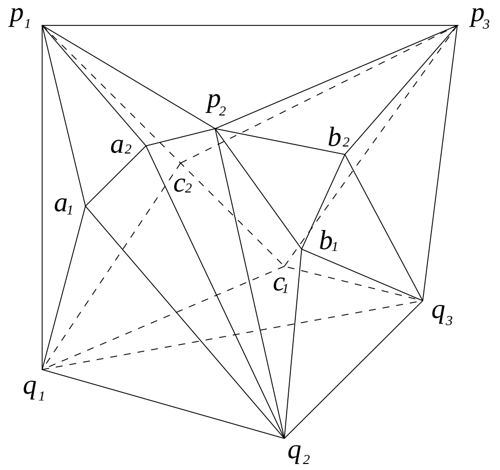


Figure 10: Schonhardt's polyhedron with three BP-nets

After removing the vertices $v_j^{(i)}$, $j = 1, \cdots, m_i - 1$, $i = 1, \cdots, k$, and their adjacent edges from $G(\mathcal{P})$, and adding edges $p^{(i)}q^{(i)}$, $i = 1, \cdots, k$, we obtain a new polyhedron $P_{h-1}$. If $P_{h-1}$ is a $(h-1)$-level BP-polyhedron, then $\mathcal{P}$ is called $h$-level BP-polyhedron, where $h$ is a positive integer. Obviously, the 1-level BP-polyhedron is a BP polyhedron. The stacked polyhedron can be considered as the 0-level BP-polyhedron. Let $\mathcal{P}'$ be a $h$-level BP-polyhedron. If $\mathcal{P}$ is constructed from $\mathcal{P}'$ by successive addition of tetrahedra on the triangle facets, then $\mathcal{P}$ is called an *augmented* $h$-level BP-polyhedron.

In Theorem 5 and 6, we proved the structure of the minimal tetrahedralization of the BP-polyhedron and the special two-level BP-polyhedron. Motivated by these results, we have the following conjecture.

**Conjecture**. If $\mathcal{P}$ is an augmented $h$-level BP-polyhedron ($h \geq 2$), then the minimal tetrahedralization of $\mathcal{P}$ consists of $E(\mathcal{P})$ and all the polar edges in $G(\mathcal{P})$ and the generated polar edges in $G(\mathcal{P}_i)$, $1 \leq i \leq h-1$.

By applying a similar method to that used in the proof of Theorem 5 and 6, we can prove most cases of this conjecture; however, the proofs are too lengthy to include in this paper, so we will present them in another paper.

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