# The Medial Axis of a Union of Balls

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## Abstract

We present an algorithm for computing the exact interior medial axis of a union of balls in $\mathbb{R}^d$. Our algorithm combines the simple characterization of this medial axis given by Attali and Montanvert with the combinatorial information provided by Edelsbrunner's $\alpha$-shape. This leads to a very simple algorithm, which we implement in $\mathbb{R}^3$.

## 1 Introduction

There is considerable interest, in mesh generation, computer graphics, computer vision and medical imaging, in describing the shape of an object by its medial axis. But computing the medial axis of a three-dimensional object is difficult in general; it can be done for simple polyhedra using exact arithmetic and computational algebra [4].

Attali and Montanvert [1] showed that, in contrast, the medial axis of a three-dimensional *union of balls* has a simple structure. A union of balls, as the name suggests, is the union $\mathcal{U}$ of a set of balls $\mathcal{B} = \{B_1, B_2 \ldots B_n\}$ in $\mathbb{R}^d$. This is an important special case, since *any* three-dimensional shape can be approximated by a finite union of balls; we recently showed [5] that given a certain well-defined sample of points $P$ on an object surface, the union of a subset of the Voronoi balls of $P$ is a good approximation of the original object, for which the errors in position and normal on the surface are everywhere bounded. Thus, given a good sample from an object surface, we can compute the exact medial axis of a nearby union of balls. In the (frequent) case in which the original object is known only by the sample points, this is as good an approximation of the medial axis as we are likely to get.

The elegant characterization of the medial axis of a union of balls given by Attali and Montanvert does not lead immediately to an algorithm. Using the beautiful duality relationship between a union of balls and its $\alpha$-*shape*, developed by Edelsbrunner [3], we give a more combinatorial and algorithmic version of this characterization. We have implemented the resulting construction in $\mathbb{R}^3$, using the `CGAL` library.

## 2 Definitions

We describe a $d$-ball $b$ by its center $p$ and its radius $\rho$. Let $\mathcal{B}$ be a set of $d$-balls in $\mathbb{R}^d$, and let $\mathcal{U}$ be their union. We assume that the balls are in general position. This simplifies the following:

**Definition:** A *k-face* of $\mathcal{U}$ is a subset of the boundary $\delta\mathcal{U}$ contained in a set of balls of cardinality exactly $(d-k)$.

Note that a face need not be connected.

**Definition:** The *vertices* of $\mathcal{U}$ are its 0-faces. Let $\mathcal{V}$ be the set of vertices of $\mathcal{U}$.

Edelsbrunner [3] studied the dual shape of $\mathcal{U}$, the $\alpha$-*shape* $\mathcal{S}$. For any collection $\mathcal{Q}$ of sets the *nerve* $N(\mathcal{Q})$ is the set of all subsets of $\mathcal{Q}$ that have a non empty intersection.

$$N(\mathcal{Q}) = \{X \subseteq \mathcal{Q} | \cap_{x \in X} \neq \phi\}$$

The set $\mathcal{B}$ of balls can be thought of as a set of weighted points. These induce a *weighted Voronoi diagram* $Vor(\mathcal{B})$, or *power diagram.* For each ball $b \in B$ let $V_b$ represent the Voronoi cell of $b$. Restricting $Vor(\mathcal{B})$ to $\mathcal{U}$ forms a partition of $\mathcal{U}$. Let $\mathcal{Q}$ be the collection of closed $d$-dimensional cells in this partition, each a subset of a cell in the weighted Voronoi diagram. That is, $\mathcal{Q} = \{V_b \cap b | b \in \mathcal{B}\}$. The $\alpha$-complex $\mathcal{K}$ is a geometric realization of the nerve of $\mathcal{Q}$, as follows. We use the notation $\sigma_T$ be represent the convex hull of the centers of balls in $T$. The $\alpha$-complex is a set of geometric simplices

$$\mathcal{K} = \{\sigma_T | T \in N(\mathcal{Q})\}$$

It turns out that the $\alpha$-*complex* is a (geometric) simplicial complex. The underlying space of this complex (ie. the union of all its simplices) is the $\alpha$-*shape* $\mathcal{S}$; see Figures 2a and 2b.

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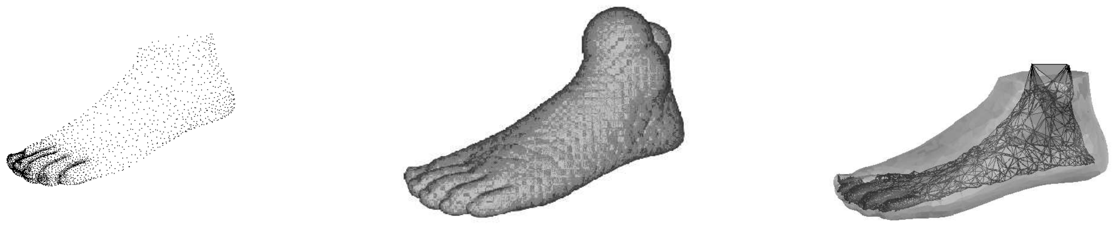

Figure 1-a: Figure 1-b: Figure 1-c:

Figure 1: Left, set of sample points $P$ on an object surface; Union of a subset of the Voronoi balls of $P$ which accurately represents the object is in the middle. Right, the medial axis of the union shown inside the object surface.

Edelsbrunner showed that the structure of the $\alpha$-shape reflects the combinatorial structure of $\mathcal{U}$. Each $k$-simplex $\sigma_T$ on the boundary $\delta\mathcal{S}$ of the $\alpha$-shape has a dual $(d-k)$-face $s_T$ on the boundary $\delta\mathcal{U}$.

We will need to distinguish the full-dimensional parts of the $\alpha$-shape from the rest.

**Definition:** A face in the boundary $\delta\mathcal{S}$ of the $\alpha$-shape is *singular* if it not a proper face of any $d$-dimensional simplex in the $\alpha$-complex $\mathcal{K}$. Let $\mathcal{M}$ be the union of the singular faces. We call the maximal connected components of $\mathcal{S}-\mathcal{M}$ the *regular components* of $\mathcal{S}$.

**Definition:** The *interior medial axis* (or simply the *medial axis*) of an object $O$ is the closure of the set of points $m \in O$ such that $m$ has at least two closest points on the surface $\delta O$.

We can now state the elegant characterization of the medial axis of a union of balls given by Attali and Montanvert.

**Theorem 1** *[1] The medial axis of a union $\mathcal{U}$ of balls consists of*
*(i)the singular faces of $\mathcal{S}$, and*
*(ii) the subset of the Voronoi diagram $Vor(\mathcal{V})$ whose closest point on the boundary $\delta\mathcal{U}$ is a vertex in $\mathcal{V}$.*

The difficulty with using this theorem directly in the computation of the medial axis $\mathcal{X}$ of $\mathcal{U}$ is that in $\mathbb{R}^3$, unlike $\mathbb{R}^2$, the Voronoi diagram of $\mathcal{V}$ can be quite complicated, including Voronoi vertices interior to $\mathcal{U}$ but not belonging to $\mathcal{X}$, and one- and two-dimensional faces which partially belong to $\mathcal{X}$. Testing which faces and parts of faces have a closest point in $\mathcal{V}$ is non-trivial and subject to numerical errors. These difficulties motivate our more combinatorial characterization.

# 3 Geometric theorems

First, we note that points in $\mathcal{U}-\mathcal{S}$ cannot belong to the medial axis of $\mathcal{U}$.

**Observation 2** *For any point $x \in \mathcal{U}-\mathcal{S}$, there is a unique point $u \in \delta\mathcal{U}$ such that for any $v, v \in \mathcal{U}$, $d(u,x) < d(v,x)$.*

Also, points in a simplex $\sigma_T$ on the surface of the $\alpha$-shape are closest to the dual *face* on $\delta\mathcal{U}$.

**Observation 3** *If $y \in \sigma_T$, where $\sigma_T$ is a simplex in $\delta\mathcal{S}$, then $y$ is the center of a $d$-ball $B_y$ with $s_T \subseteq B_y \subseteq \mathcal{U}$.*

**Lemma 4** *Let $\mathcal{C}$ be a regular component of $\mathcal{S}$. Then for any point $x \in \mathcal{C}$, either*
*a) $x$ has exactly one closest point $p$ on $\delta\mathcal{U}$, and $p$ is a vertex of $\mathcal{U}$, or*
*b) $x$ has at least two closest points $p, q \in \delta\mathcal{U}$ which are vertices of $\mathcal{U}$.*

**Proof:** We first consider any point $x$ in the interior of $\mathcal{C}$. We claim that any closest point to $x$ must be a vertex. Assume for the purpose of contradiction that there is a point $p$ closest to $x$ such that $p \in s_T$ for some face $s_T$ of dimension greater than zero. Consider the ball $B$ centered at $x$ and touching $p$. Since $p$ is a closest point to $x$, $B \subseteq \mathcal{U}$. The segment $px$ intersects the boundary of $S$ in some point $y \in \sigma_T$. By Observation 3, there is a ball $B_y$ centered at $y$ with $s_T \subseteq B_y \subseteq \mathcal{U}$, and since $y \in px$, point $p$ is also a closest surface point to $y$. Since $y$ is closer to $p$ than $x$ is, $B_y - p$ lies in the interior of $B-p$, so that the non-empty set $s_T - p$ lies in the interior of $B-p$, which contradicts $B \subseteq \mathcal{U}$.

Now consider the case in which $x \in \delta\mathcal{C}$. Let $\sigma_T$ be the simplex of smallest dimension in $\delta\mathcal{C}$ containing $x$. If $\sigma_T$ is a $(d-1)$-simplex, then the only closest point to $x$ in $\delta\mathcal{C}$

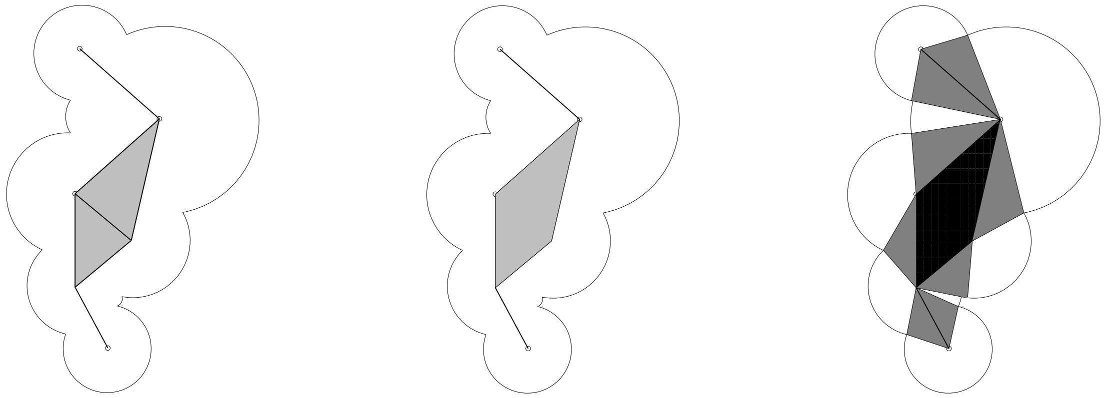

Figure 2-a: Figure 2-b: Figure 2-c:

Figure 2: Left, the $\alpha$*-complex* $\mathcal{K}$ of a union $\mathcal{U}$ of balls. The $\alpha$*-shape* $\mathcal{S}$ which is the underlying space of $\mathcal{K}$ is in the middle. Right, $\mathcal{U} - \mathcal{S}$ is covered by the grey and white regions, each connecting a face $\sigma_t$ of $\delta\mathcal{S}$ and its dual face $s_T$ on the surface $\delta\mathcal{U}$.

is the dual face $s_T$, which is a singular point. Otherwise, $\sigma_T$ is incident to at least two $(d-1)$-simplices in $\delta\mathcal{C}$, so that its dual face $s_T$ must contain at least two singular points closest to $x$.
□

Notice that the $(d-1)$-faces of $\delta\mathcal{S}$ do not belong to the medial axis of $\mathcal{U}$, but that the $(d-k)$-faces do, for $1 < k \leq d$.

**Lemma 5** *Any point belonging to a singular face of* $\mathcal{S}$ *belongs to the medial axis of* $\mathcal{U}$.

**Proof:** By Observation 3, any point $x$ contained in a face $\sigma_t$ dual to a face $s_T$ of dimension greater than zero is the center of a ball $B$ with $s_T \subseteq B \subseteq \mathcal{U}$. Since $s_T \subseteq \delta B$, $x$ is a point of the medial axis. A point $x$ contained in a singular $(d-1)$-face of the $\alpha$-shape, on the other hand, is dual to two vertices of the union $\mathcal{U}$.
□

The following theorem gives our chacterization of the medial axis of $\mathcal{U}$.

**Theorem 6** *The medial axis of a union* $\mathcal{U}$ *of balls in* $\mathbb{R}^d$ *consists of*
*(i) the singular faces of* $\mathcal{S}$*, and*
*(ii) the subset of* $Vor(\mathcal{V})$ *that intersects the regular components of* $\mathcal{S}$.

**Proof:** By Lemma 5, any point in a singular face of $\mathcal{S}$ belongs to the medial axis. By Lemma 4, any point in a regular component must have at least one closest point on the boundary which belongs to $\mathcal{V}$, and if it belongs to $Vor(\mathcal{V})$, it must be equidistant from at least two vertices in $\mathcal{V}$; hence it too must belong to the medial axis of $\mathcal{U}$.

Now consider a point $m$ in the medial axis of $\mathcal{U}$; it cannot be a point in $\mathcal{U} - \mathcal{S}$ or a point on the surface of $\mathcal{U}$, by Observation 2. If it belongs to some regular component then it must have more than one closest point $\in \mathcal{V}$, by Lemma 4. Otherwise $m$ belongs to a singular face.
□

# 4 Algorithm

This version of the chracterization leads to a combinatorial algorithm for computing the medial axis $\mathcal{X}$ of a union of balls in $\mathbb{R}^3$.

We start with the construction of the $\alpha$-shape of $\mathcal{B}$. Using the $\alpha$-shape, we compute $\mathcal{V}$, the vertices of $\delta\mathcal{U}$ (dual to the $(d-1)$-faces of $\delta S$). All the singular faces of the $\alpha$-shape belong to the medial axis. For each regular component $\mathcal{C}$, we build the Voronoi diagram of the subset of $\mathcal{V}$ dual to the traingular faces of $\delta\mathcal{C}$, and then compute the intersection of the Voronoi diagram with $\mathcal{C}$. We compute the Voronoi diagram for each regular component seperately since, when possible, it is faster to do several smaller Voronoi diagram computations rather than one big one.

Now we determine which parts of the Voronoi diagram $Vor(\mathcal{V})$ intersect $\mathcal{C}$. This is easy once we can label each Voronoi vertex as inside or outside of the regular component $\mathcal{C}$, but the labeling is non-trivial since in general $C$ is not convex. We will therefore label vertices by tracing the one-skeleton of $Vor(\mathcal{V})$.

We make some additional observations which simplfy the labeling process. Notice that any $(d-2)$-face $f$ of $\delta\mathcal{C}$ is contained in exactly two $(d-1)$-faces of $\delta\mathcal{C}$. These are dual to vertices $v_1, v_2 \in \mathcal{V}$, which are the two closest vertices in $\mathcal{V}$ to any point $x \in f$. So there is a $(d-1)$-face $h$ of the Voronoi diagram of $\mathcal{V}$ separating the Voronoi cells of $v_1$ and $v_2$, and this face is split into two parts by $f$. We call any such $h$ a *cross-face.* Cross-faces are in one-to-one correspondence with the $(d-2)$-faces of $\delta\mathcal{C}$.

Any $(d-1)$-face of $Vor(\mathcal{V})$ which crosses the boundary of $\mathcal{C}$ must intersect the boundary $\delta\mathcal{C}$ in a $(d-2)$-dimensional set $s$. Since every point in the interior of a $(d-1)$-face in $\delta\mathcal{C}$ is closest to exactly one vertex in $\mathcal{V}$, $s$ must be a $(d-2)$-face on $\delta\mathcal{C}$. This gives us the following.

**Observation 7** *The only $(d-1)$-faces of $Vor(\mathcal{V})$ that cross the boundary of a regular component $\mathcal{C}$ are cross-faces. All other $(d-1)$-faces either lie completely inside or completely outside $\mathcal{C}$.*

Thus, labeling the vertices of any $(d-1)$-face of $Vor(\mathcal{V})$ which is *not* a cross-face is easy; once we learn a label for one vertex $x$, all the other vertices of $h$ get the same label as $x$.

Labeling the vertices of a cross-face $h$ is conceptually simple. While walking on the one-skeleton of $h$, starting from $x$, we toggle the label every time we cross $f$; vertices in $f$ get labeled as inside.

To avoid numerical problems involved with testing vertices against $f$, we give a combinatorial characterization of when the one-skeleton of $h$ crosses $f$, based on the following.

**Observation 8** *An edge of the one-skeleton of $h$ intersects $f$ only at a vertex of $f$.*

This follows because for any $k$-face $g$ of $f$, of dimension $k > 0$, the set of closest vertices in $\mathcal{V}$ from any point $x \in g$ is the set $\mathcal{V}_g \subset \mathcal{V}$ of at least $d-k$ vertices which all lie in the $(d-k)$-face of $\mathcal{U}$ dual to $g$. The set $\mathcal{V}_g$ thus determines (possibly degenerately) a face of $Vor(\mathcal{V})$ containing $g$, and hence of dimension at least $k$. This means that the interior of $g$ cannot intersect an edge of $Vor(\mathcal{V})$.

All we have to do, then, is to locate the vertices of $f$ on the one-skeleton of $h$; some will coincide with vertices of $h$ and others will lie on edges. These vertices are all labeled as inisde; the others vertices of $h$ can be labeled by traversing the one-skeleton.

We summarize our algorithm for extracting the medial axis of a union of balls in $\mathbb{R}^3$.

**In:** A set $\mathcal{B}$ of balls and their $\alpha$-shape $\mathcal{S}$.

**Out:** The medial axis $X$ of the union $\mathcal{U}$ of $\mathcal{B}$.

**1:** Compute the set of vertices $\mathcal{V}$ of $\mathcal{U}$ as the duals of the triangles of $\delta\mathcal{S}$.
**2:** Add all edges (singular and regular) of $\mathcal{S}$ to $X$.
**3:** Add all singular triangles in $\mathcal{S}$ to $X$.
**4:** For each regular component $C$ of $\mathcal{S}$ :
- **a:** Construct the Voronoi diagram of the subset of $\mathcal{V}$ dual to triangles in $\delta\mathcal{C}$.
- **b:** Split every center edge in the Voronoi diagram at the ball center.
- **c:** Split every Voronoi face containing an edge of $\mathcal{S}$ connecting two ball centers.
- **d:** Label a Voronoi vertex at infinity as exterior.
- **e:** Label each Voronoi face $h$ with at least one labeled vertex $x$ as follows:
  - (i) If $h$ is not a cross face , all vertices of $h$ get the label of $x$.
  - (ii) If $h$ is a cross face that passes through an edge $uv$ of $\mathcal{C}$, $uv$ splits $h$ into two parts. All vertices in the part of $h$ containing $x$ get the label of $x$ and the remaining vertices get the opposite label.
- **f:** Add to $\mathcal{X}$ all faces of the Voronoi diagram whose vertices all lie in $\mathcal{S}$.

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