1-bend 3-D orthogonal drawings: two open problems solved

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Abstract

In this paper, we study three-dimensional orthogonal drawings which have at most one bend per edge. We prove a number of lower bounds and impossibility-results for such drawings, settling open problems posed in two previous papers on the same subject.

1 Background

A *3-D orthogonal drawing* of a graph is a representation of the graph where vertices are represented by boxes and edges are represented as disjoint routes along an underlying three-dimensional rectangular grid. (As we will not be studying any other type of drawings, we drop "3-D" and "orthogonal" from now on.) For the remainder, we identify the graph-theoretic object (vertex, edge) and its representation in a drawing.

Each edge consists of axis-parallel segments. The transition from one segment to another is called a *bend*. In this paper, we study drawings with very few bends per edge. We say that a drawing is a *k-bend drawing* if all edge routes have at most k bends.

For any vertex v, let $X(v)$ be the number of different integral x-coordinates of the box of v; similarly we define $Y(v)$ and $Z(v)$. The *surface of* v is $2(X(v)Y(v) + Y(v)Z(v) + Z(v)X(v))$. The *volume of* v is defined as $X(v)Y(v)Z(v)$.

Given a drawing, we denote by $X \times Y \times Z$ the size of the smallest enclosing rectangular box of the drawing. The *volume* of the drawing is $X \cdot Y \cdot Z$.

1.1 Existing results

In this paper, we study volume bounds of 1-bend three-dimensional drawings. We chose 1-bend drawings, because not every graph has a 0-bend drawing (also known as visibility representation) [BSWW99, FM99].

In [BSWW99], it was shown that the complete graph K_n has a 1-bend drawing with $O(n^3)$ volume (more precisely, in an $n/2 \times n/2 \times n/2$-grid.) In the same paper, it was also shown that any drawing of K_n has volume $\Omega(n^{2.5})$. However, the lower bound does not take restrictions on the number of bends into account, and in particular, it was left as an open problem whether any 1-bend drawing of K_n needs $\Omega(n^3)$ volume.

One criticism of the drawings in [BSWW99] is that the vertices resemble "sticks", i.e., one dimension of the vertices is very large while the other two are 1. It would be more visually appealing if vertices resembled cubes. More precisely, we say that a drawing has *bounded aspect-ratios* if there exists a constant $r \geq 1$ such that any vertex v has aspect ratio $\leq r$. If $r = 1$, then the drawing is a *cube-drawing.*

The construction in [BSWW99] can be converted into a cube-drawing by "blowing up" every vertex (see also Figure 2). However, this increases the volume of the drawing to $O(n^4)$. Also, the surface of each vertex then becomes $O(n^2)$, which seems excessive since every vertex has only $O(n)$ incident edges. We would prefer a drawing where the surface of each vertex is proportional to its degree. More precisely, we say that a drawing is *degree-restricted* if the surface of a vertex v is at most $\alpha\, deg(v)$, for some constant $\alpha \geq 1$.

The construction in [BSWW99] is degree-restricted for K_n, but when converted to a cube-drawing, it is no longer degree-restricted. Hence, the question was posed whether every graph has a degree-restricted 1-bend cube-drawing.

In [BTW00], the lower bounds of [BSWW99] were extended to graphs with arbitrarily many edges. More precisely, it was shown that there exist graphs with n vertices and m edges that have volume $\Omega(mn^{1/2})$ in any drawing. This lower bound also does not take restrictions on the number of bends into account.

Finally, in [Woo00], it was shown that every n-vertex m-edge graph with genus g has a 1-bend drawing of volume $O(nm\sqrt{g})$, which is $O(nm^{3/2})$ in the worst case.

1.2 Our results

In this paper, we settle the two open problems mentioned above, and provide a variety of related results. Specifically, we show the following:

- Let Δ be the maximum degree of a graph. Any 1-bend cube-drawing of a graph G with $\Omega(\Delta n)$ edges represents $\Omega(n)$ many vertices with an $\Omega(\Delta) \times \Omega(\Delta) \times \Omega(\Delta)$-box.

  This has the following consequences:

  – G does not have a degree-restricted cube-

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drawing, which settles the second open problem mentioned above.

- Any 1-bend cube-drawing of G has volume $\Omega(\Delta^3 n)$.
- Since K_n is $(n-1)$-regular, in particular any 1-bend cube-drawing of K_n has volume $\Omega(n^4)$. [This bound is matched by a construction.]

- Other lower bounds are provided by using d-regular n-vertex graphs $G_{n,d}$ with special cut-properties. We show the following:
  - Any 1-bend drawing of $G_{n,d}$, for n and d sufficiently big, has at least $\frac{1}{8}n$ vertices that are all intersected by one grid plane.
  - Any 1-bend drawing of $G_{n,d}$ has volume $\Omega(n^2 d)$.

    Since $K_n = G_{n,n-1}$, any 1-bend drawing of K_n has volume $\Omega(n^3)$, which settles the first open problem mentioned above.

2 Cube-drawings

We first address the question whether there exist degree-restricted 1-bend cube-drawings. The answer is to the negative, and we prove this by showing that in any 1-bend drawing, a lot of vertices are intersected by a lot of grid planes, at least if the number of edges is big.

Lemma 2.1 *If G is a simple graph with $\Omega(\Delta n)$ edges, then in any 1-bend drawing of G, $\Omega(n)$ vertices are intersected by $\Omega(\Delta)$ grid planes each. More precisely, if G has $\geq \kappa\Delta n$ edges, for some $\kappa > 0$, then $\geq \frac{1}{6}\kappa n$ vertices are intersected by $\geq \frac{1}{6}\kappa\Delta$ grid planes each.*

Proof: Since G is a 1-bend drawing, any edge e is entirely contained within one plane P. We say that edge e *belongs* to P and P *owns* e. (If e has no bend, then it is contained in two planes. We arbitrarily choose one of those planes to own e, so that each edge belongs to exactly one plane.)

Let $P_1, \ldots, P_l$ be the planes that own at least one edge. For $i = 1, \ldots, l$, let $n(P_i)$ be the number of vertices in P_i that are incident to one edge owned by P_i. (Note that there might well be vertices intersected by P_i that do not own an edge in P_i; these vertices do not contribute to $n(P_i)$.) See also Figure 1.

The crucial observation is that edges do not cross, hence the graph formed by the edges owned by P_i is planar. In particular, by simplicity at most $3n(P_i)$ edges can be owned by P_i. Therefore,

$$\sum_{i=1}^{l} 3n(P_i) \geq \#\text{ edges} \geq \kappa\Delta n.$$

Now we count $\sum_{i=1}^{l} n(P_i)$ in another way. For every vertex v, denote by $p(v)$ the number of planes that own an incident edge of v; see also Figure 1. Then $\sum_{i=1}^{l} n(P_i) = \sum_{v \in V} p(v)$, because both sums count the incidences between vertices and planes owning incident edges of the vertices. We use this to obtain a lower bound on $p(v)$ for $\Omega(n)$ many vertices; by $p(v) \leq X(v) + Y(v) + Z(v)$ this yields the result.

Let V_b be the vertices with $p(v) \geq \frac{1}{6}\kappa\Delta$, and assume to the contrary that fewer than $\frac{1}{6}\kappa n$ vertices belong to this set. Observe that $p(v) \leq \Delta$ for all vertices (because for each plane there is at least one incident edge of v), and that at most n vertices could be in $V - V_b$. We have

$$\begin{aligned}
\sum_{i=1}^{l} n(P_i) &= \sum_{v \in V} p(v) = \sum_{v \notin V_b} p(v) + \sum_{v \in V_b} p(v) \\
&< |V_s| \cdot \frac{1}{6}\kappa\Delta + |V_b| \cdot \Delta \\
&< n \cdot \frac{1}{6}\kappa\Delta + \frac{1}{6}\kappa n \cdot \Delta = \frac{1}{3}\kappa\Delta n.
\end{aligned}$$

This contradicts the previous inequality. $\square$

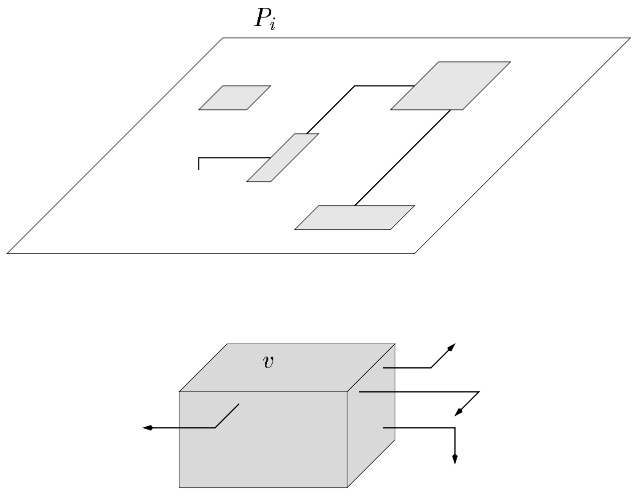

Figure 1: Illustration of $n(P_i)$ and $p(v)$. In these examples, $n(P_i) = 3$ and $p(v) = 2$.

Two remarks are in order. First, note that the constants in the above lemma could be improved in case G is bipartite, because then at most $2n_i$ edges could be within one grid plane P_i. While this would improve some of the lower bounds to follow by a small fraction, we will not pursue this detail for simplicity's sake. Second, note that all we needed for the proof is that any edge is routed entirely within one plane. While this certainly holds for any 1-bend drawing, it also holds for many other constructions (e.g., the ones of [BSWW99]). Hence, the lemma and its corollaries could be generalized to any so-called *co-planar* drawing in which each edge is routed within a plane.

Using the above lemma, we obtain that in any 1-bend cube-drawing many vertices must have big boxes. In fact, this result holds for any drawing with bounded aspect ratios.

Lemma 2.2 *Let G be a graph with $\Omega(\Delta n)$ edges. Then any 1-bend drawing of G with aspect ratios $\leq r$ contains $\Omega(n)$ vertices with minimum dimension $\Omega(\Delta/r)$, surface $\Omega(\Delta^2/r)$ and volume $\Omega(\Delta^3/r^2)$.*

More precisely, if G has $\geq \kappa\Delta n$ edges for some $\kappa > 0$, then $\geq \frac{1}{6}\kappa n$ vertices have minimum dimension $\geq \frac{1}{18}\kappa\Delta/r$, surface $\geq \frac{1}{81}\kappa^2\Delta^2/r$ and volume $\geq \frac{1}{5832}\kappa^3\Delta^3/r^2$.

Proof: Let v be one of the $\geq \frac{1}{6}\kappa n$ vertices that is intersected by at least $\frac{1}{6}\kappa\Delta$ grid planes. Let the box representing v be an $X \times Y \times Z$-box; without loss of generality we may assume $X \leq Y \leq Z$. By the condition on v, we must have $X + Y + Z \geq \frac{1}{6}\kappa\Delta$. Also, by the aspect ratio, we must have $Z \leq rX$.

The minimum dimension of v is X. Minimizing X under the above constraints yields $X \geq \frac{1}{6}\kappa\Delta/(1+2r) \geq \frac{1}{18}\kappa\Delta/r$ by $r \geq 1$.

The surface of v is $2(XY + YZ + XZ)$ and the volume is XYZ. Minimizing each expression, subject to the above constraints, one obtains (for both of them) the solution $X = \frac{1}{6}\kappa\Delta/(r+2) = Y$ and $Z = rX = \frac{1}{6}\kappa r\Delta/(r+2)$. Hence the surface of v is at least $\frac{2}{36}\kappa^2\Delta^2(1+2r)/(r+2)^2 \geq \frac{1}{18}\kappa^2\Delta^2 2r/(3r)^2 = \frac{1}{81}\kappa^2\Delta^2/r$, and the volume is at least $\frac{1}{216}\kappa^3\Delta^3 r/(r+2)^3 \geq \frac{1}{216}\kappa^3\Delta^3 r/(3r)^3 = \frac{1}{5832}\kappa^3\Delta^3/r^2$. □

As a corollary we obtain the desired impossibility result for any graph G with $\Omega(\Delta n)$ edges: a degree-restricted drawing of G with bounded aspect ratios is impossible, because $\Omega(n)$ vertices have surface $\Omega(\Delta^2/r)$. In fact, any drawing with aspect ratios in $o(\Delta)$ are impossible. Note that many such graphs G exist, for example, any Δ-regular n-vertex graph has $m = \frac{1}{2}\Delta n$ edges.

Theorem 1 *Any G with $\Omega(\Delta n)$ edges has no degree-restricted 1-bend drawing with aspect ratios $o(\Delta)$.*

As another corollary, we obtain a bound on the volume of drawings with bounded aspect ratios. We showed that for any graph with $\Omega(\Delta n)$ edges, we have $\Omega(n)$ vertices with volume $\Omega(\Delta^3/r^2)$ in any cube-drawing with aspect ratios $\leq r$. Adding these volumes, we obtain the following bound:

Theorem 2 *If G has $\Omega(\Delta n)$ edges, then any 1-bend drawing with aspect ratios $\leq r$ has volume $\Omega(\Delta^3 n/r^2)$.*

Depending on the values of Δ and r, this improves in some cases on the lower bound of $\Omega(m^{3/2}/\sqrt{r})$ for such drawings known from [BTW00].

The above lower bound is optimal for cube-drawings K_n, because the lower bound states $\Omega(n^4)$ for K_n, and a construction with volume $O(n^4)$ can be obtained easily by "blowing up" the vertices of the construction of [BSWW99]. See Figure 2.

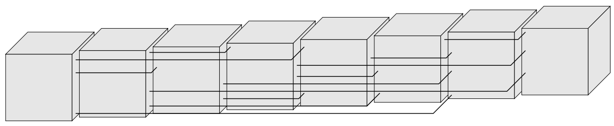

Figure 2: A cube-drawing of K_n with volume $O(n^4)$. Only half of the edges are shown; the other half is routed behind the cubes.

3 Lower Bounds

Now we prove lower bounds on the volume of 1-bend drawings. We know that $O(n^3)$ volume is possible for K_n [BSWW99], and we will show that this is optimal.

Our lower bound proof follows a scheme developed in [BSWW99] and also used in [BTW00]. Given a drawing, we consider three cases: (1) One grid line intersects "many" vertices; (2) one grid plane, but no grid line, intersects "many" vertices; (3) neither of the above is the case. In [BSWW99], it was shown that the volume of K_n is $\Omega(n^3)$ in the first and third case, but in the second case, only a bound of $\Omega(n^{2.5})$ was achieved. In [BTW00], it was shown that the volume for so-called Ramanujan-graphs is $\Omega(\Delta n^2)$ in the first case, $\Omega(\Delta n^{1.5})$ in the second case and $\Omega((\Delta n)^{1.5})$ in the third case.

We aim here for a lower bound of $\Omega(\Delta n^2)$ for all 1-bend drawings of Ramanujan-graphs. If the drawing is in the first case, then this follows directly from [BTW00] (we repeat the proof here for completeness). We prove the bound for the second case by using that every edge has at most 1 bend, and hence the two endpoints must "see" each other in some sense. We did not succeed in proving the bound for the third case, but realized that the third case cannot happen for sufficiently large n when edges have at most one bend.

This section is structured as follows: We first review the Ramanujan-graphs. Then we prove that the third case cannot happen. Finally, we proceed to prove lower bounds for all drawings.

3.1 Ramanujan-graphs

We first study the so-called *Ramanujan graphs*, which were introduced in [LPS88] and have already been used in [BTW00] to provide lower bounds for orthogonal graph drawing. They have the useful property that for any two disjoint subsets of size $\Omega(n)$, there are $\Omega(m)$ edges between the two subsets. More precisely:

Lemma 3.1 *Let $0 < \mu < 1$ be a constant. If $p \neq q$ are primes, $p \equiv 1 \bmod 4$, $q \equiv 1 \bmod 4$, $p+1 \geq 16/\mu^2$, then there exists a simple graph $G_{n,d}$ (called* Ramanujan-graph*) with the following properties:*

- *$G_{n,d}$ is d-regular for $d = p + 1$,*
- *the number n of vertices of $G_{n,d}$ is at least $q(q-1)/2$ and at most $q(q-1)$.*
- *for any disjoint vertex sets S, T of $G_{n,d}$ with $|S| \geq \mu n$, $|T| \geq \mu n$, there are at least $\frac{1}{2}\mu^2 \cdot dn$ edges between S and T.*

Proof: The proof is identical to the one in [BTW00], except that we replace "1/6" in their proof with μ in ours.

Namely, let $G_{n,d}$ be the graph $X^{p,q}$ defined in [LPS88]; the first two properties of the graph were shown in this paper. It was also shown that $\lambda \leq 2\sqrt{d-1}$, where λ denotes the second-largest eigenvalue of $G_{n,d}$. Assume S and T are as specified above. We know from [AS92] that the number of edges between S and T is at least $\frac{d|S||T|}{n} - \lambda\sqrt{|S||T|}$. Now,

$$\begin{aligned}\lambda\sqrt{|S||T|} &\leq 2\sqrt{d-1}\sqrt{|S||T|}\underbrace{\sqrt{|S||T|/\mu^2 n^2}}_{\geq 1}\underbrace{\sqrt{d\mu^2/16}}_{\geq 1} \\ &\leq \frac{1}{2}d|S||T|/n.\end{aligned}$$

Hence, the number of edges between S and T is at least $(1-\frac{1}{2}) \cdot d|S||T|/n \geq \frac{1}{2}\mu^2 \cdot dn$. □

In this paper, we will state the lower bounds only for Ramanujan-graphs. This is not a big restriction, because as was shown in [BTW00], graphs containing Ramanujan-graphs can be constructed for almost all values of m and n.

Lemma 3.2 *[BTW00] There exist constants n_0 and d_0 such that for any $n \geq n_0$ and any $m \geq d_0 n$ there exists a graph with n vertices and m edges that has a Ramanujan-graph with $\theta(n)$ vertices and $\theta(m)$ edges as subgraph.*

In particular, using these graphs, the lower bounds can be transferred from Ramanujan-graphs to all values of n and m without affecting the order of magnitude. See [BTW00] for details.

3.2 Vertices in one plane

Now we prove that the "third case" mentioned above cannot happen for 1-bend drawings of Ramanujan-graphs, i.e., there always exists a grid plane intersecting $\Omega(n)$ vertices. For this and the lower bound proofs to come, we will often refer to positions of vertices relative to planes. We say that a vertex is *left* of an $(X = X_0)$-plane if all the points in its grid box have X-coordinates less than X_0. The notion of *right* of an $(X = X_0)$-plane is analogous. We say that a vertex is *before* an $(Y = Y_0)$-plane if all the points in its grid box have Y-coordinates less than Y_0. The notion of *behind* an $(Y = Y_0)$-plane is analogous. We say that a vertex is *below* an $(Z = Z_0)$-plane if all the points in its grid box have Z-coordinates less than Z_0. The notion of *above* an $(Z = Z_0)$-plane is analogous.

Also, for the proofs to come, for ease of notation we neglect rounding issues, and assume that n is divisible as needed. This has no effect on the order of magnitude of the lower bounds, since for example in the next theorem, one could show a bound of $\frac{1}{8}n - o(n)$ vertices for all values of n.

Theorem 3 *Let $G_{n,d}$ be a Ramanujan-graph with $d \geq 2^{16}$ and n divisible by 8. Then any 1-bend drawing of $G_{n,d}$ has a grid plane that intersects at least $\frac{1}{8}n$ vertices.*

Proof: Assume to the contrary that no grid plane intersects as many as $\frac{1}{8}n$ vertices.

Intuitively, the claim holds because the drawing can be split into non-empty octants. Two of these octants have no grid-plane in common, and hence cannot have an edge between them. See Figure 3 for an illustration. The precise proof is as follows:

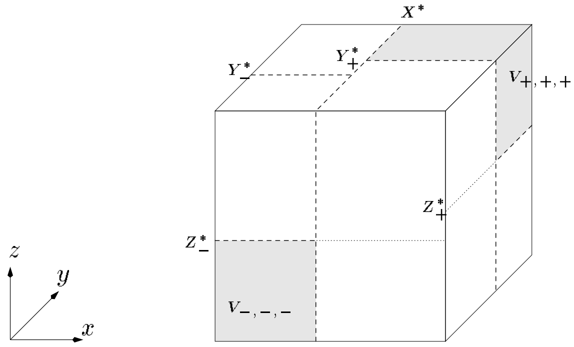

Figure 3: By splitting the drawing into octants, we can find two non-empty sets of vertices that cannot have an edge with ≤ 1 bend connecting them.

As an $(X = X_0)$-plane is swept from smaller to larger values, we encounter an integer X' where, for the last time, there are at most $\frac{7}{16}n$ vertices to the left of the $(X = X')$-plane. Thus, there are at least $\frac{7}{16}n$ vertices to the left of the $(X = X'+1)$-plane. All these vertices, which we denote V_-, are also to the left of the $(X = X'+\frac{1}{2})$-plane. Also, since the $(X = X')$-plane intersects at most $\frac{1}{8}n$ vertices, and at most $\frac{7}{16}n$ vertices are to the left of it, there are at least $n - \frac{1}{8}n - \frac{7}{16}n = \frac{7}{16}n$ vertices to the right of the $(X = X')$-plane. All these vertices, which we denote V_+, are also to the right of the $(X = X'+\frac{1}{2})$-plane. Denote $X^* = X' + \frac{1}{2}$.

Note that no X-plane intersects both a vertex in V_+ and a vertex in V_-.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_-; recall that $|V_-| \geq \frac{7}{16}n$. We thus obtain a value Y_-^* such that at least $\frac{5}{32}n$ vertices of V_- are before the $(Y = Y_-^*)$-plane, and at least $\frac{7}{16}n - \frac{1}{8}n - \frac{5}{32}n = \frac{5}{32}n$ vertices of V_- are behind the $(Y = Y_-^*)$-plane. Denote these two sets of vertices as $V_{-,-}$ and $V_{-,+}$.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_+. We thus obtain a value Y_+^* such that at least $\frac{5}{32}n$ vertices of V_+ are before the $(Y = Y_+^*)$-plane, and at least $\frac{5}{32}n$ vertices of V_+ are behind the $(Y = Y_+^*)$-plane. Denote these two sets of vertices as $V_{+,-}$ and $V_{+,+}$.

Without loss of generality, we may assume that $Y_-^* \leq Y_+^*$. In particular therefore, no Y-plane intersects both a vertex in $V_{-,-}$ and a vertex in $V_{+,+}$.

Apply the same argument to a sweep with a $(Z = Z_0)$-plane, considering only the vertices in $V_{-,-}$; recall that $|V_{-,-}| \geq \frac{5}{32}n$. We thus obtain a value Z_-^* such that at least $\frac{1}{64}n$ vertices of $V_{-,-}$ are below the $(Z = Z_-^*)$-plane, and at least $\frac{5}{32}n - \frac{1}{8}n - \frac{1}{64}n = \frac{1}{64}n$ vertices of $V_{-,-}$ are above the $(Z = Z_-^*)$-plane. Denote these two sets of vertices as $V_{-,-,-}$ and $V_{-,-,+}$.

Apply the same argument to a sweep with a $(Z = Z_0)$-plane, considering only the vertices in $V_{+,+}$. We thus obtain a value Z_+^* such that at least $\frac{1}{64}n$ vertices of $V_{+,+}$ are below the $(Z = Z_+^*)$-plane, and at least $\frac{1}{64}n$ vertices of $V_{+,+}$ are above the $(Z = Z_+^*)$-plane. Denote these two sets of vertices as $V_{+,+,-}$ and $V_{+,+,+}$.

Without loss of generality, we may assume that $Z_-^* \leq Z_+^*$. In particular therefore, no Z-plane intersects both a vertex in $V_{-,-,-}$ and a vertex in $V_{+,+,+}$.

Hence we have now shown that no grid plane intersects both a vertex in $V_{-,-,-}$ and $V_{+,+,+}$. These sets each contain at least $\frac{1}{64}n$ vertices. Since $G_{n,d}$ is a Ramanujan-graph with $d \geq 2^{16} = 16 \cdot 64^2$, we have at least $1/2^{-13} \cdot dn \geq 1$ edges between these two vertex sets. These edges cannot be drawn with ≤ 1 bend, a contradiction. □

Remark: Any constant $< 1/7$ could take the role of $1/8$ in the theorem; the smaller the constant, the smaller also the lower bound on d. For example, if we replaced $1/8$ by $1/407$, then one can compute that $d \geq 82$ suffices.

Also note that the above proof did not use that the drawing had no crossings, and hence would hold even if crossings were allowed.

3.3 1-bend drawings

Now we prove that any 1-bend drawing must have a large volume. The constants in the proof to follow are rather small and chosen for the convenience of a simple proof; they could likely be improved with a more careful analysis.

Theorem 4 *Let $G_{n,d}$ be a Ramanujan-graph with $d \geq 2^{22}$ and n divisible by 512. Then any 1-bend drawing of $G_{n,d}$ has volume $\geq 2^{-27}dn^2$.*

Proof: We distinguish two cases:

Case 1: There exists a grid line that intersects at least $\frac{1}{256}n = 2^{-8}n$ many vertices. Assume that this grid line is an X-*line*, i.e., a grid line parallel to the X-axis; the other two cases are similar.

The argument in this case is exactly the same (except for a change of constants and directions) as in [BTW00]. Namely, let $v_1, \ldots, v_t$ be the vertices intersected by the X-line, listed in order of occurrence along the line. Let X_0 be a not necessarily integer X-coordinate such that the $(X = X_0)$-plane intersects none of these t vertices and separates the first $2^{-9}n$ of them from the remaining ones, of which there are at least $2^{-9}n$ many.

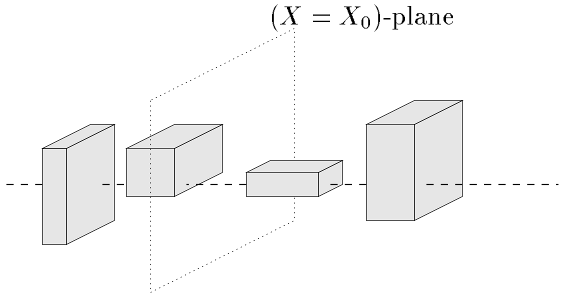

Figure 4: Case 1: many vertices intersected by one grid line.

Because $G_{n,d}$ is a Ramanujan-graph and $d \geq 16 \cdot 2^18$, at least $2^{-19} \cdot dn$ edges connect these two groups. These edges cross the $(X = X_0)$-plane, which thus must contain at least $2^{-19} \cdot dn$ points having integer Y- and Z-coordinates. Hence $YZ \geq 2^{-19} \cdot dn$. Since the X-line intersects at least $2^{-8}n$ vertices, we have $X \geq 2^{-8}n$, so $XYZ \geq 2^{-27} \cdot dn$.

Case 2: No grid line intersects many vertices.

By Theorem 3, there exists a grid plane, say the $(Z = Z')$-plane, that intersects at least $\frac{1}{8}n$ vertices; denote these vertices as V'. In all of the following argument, we will only consider vertices of V'.

As an $(X = X_0)$-plane is being swept from smaller to larger values, the intersection of the $(X = X_0)$-plane with the $(Z = Z')$-plane is a Y-line, which by assumption intersects at most $\frac{1}{256}n$ vertices at any one time. With an argument similar as in the proof of Theorem 3, we can thus obtain a value X^* such that at least $\frac{15}{256}n$ vertices of V' are to the left of the $(X = X^*)$-plane, and at least $\frac{1}{8}n - \frac{1}{256}n - \frac{15}{256}n \geq \frac{15}{256}n$ vertices of V' are to the right of the $(X = X^*)$-plane. Denote these two sets of vertices as V_- and V_+. Note that no X-plane intersects both a vertex in V_+ and a vertex in V_-.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_-. We thus obtain a value Y^*_- such that at least $\frac{7}{256}n$ vertices of V_- are before the $(Y = Y^*_-)$-plane, and at least $\frac{15}{256}n - \frac{1}{256}n - \frac{7}{256}n = \frac{7}{256}n$ vertices of V_- are behind the $(Y = Y^*_-)$-plane. Denote these two sets of vertices as $V_{-,-}$ and $V_{-,+}$.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_+. We thus obtain a value Y^*_+ such that at least $\frac{7}{256}n$ vertices of V_+ are before the $(Y = Y^*_+)$-plane, and at least $\frac{7}{256}n$ vertices of V_+ are behind the $(Y = Y^*_+)$-plane. Denote these two sets of vertices as $V_{+,-}$ and $V_{+,+}$.

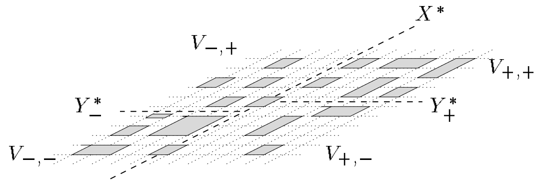

Figure 5: Case 2: many vertices intersected by one grid plane.

Without loss of generality, we may assume that $Y^*_- \leq Y^*_+$. In particular therefore, no Y-plane intersects both a vertex in $V_{-,-}$ and a vertex in $V_{+,+}$.

Since we have a Ramanujan-graph, there must be edges between $V_{-,-}$ and $V_{+,+}$ (which each have $\frac{7}{256}n$ vertices). None of these edges can be routed within an X-plane or a Y-plane as observed above, hence they are all routed within a Z-plane.

Now we use the fact that every edge is drawn with one bend. Namely, let (v, w) be an edge with $v \in V_{-,-}$ and $w \in V_{+,+}$, and assume that it is routed in the $(Z = z)$-plane. Then this edge consists of one X-segment and one Y-segment. If (say) it has an X-segment incident to v, then no other vertex can be placed on this line between v and the $(X = X^*)$-plane. This motivates the following definition illustrated in Figure 6.

Definition 1 *We say that a vertex $v \in V_{-,-}$ ($v \in V_{+,+}$) is* exposed at level z *if*

- *there exists an X-line with integer coordinates in the $(Z = z)$-plane that intersects v and does not intersect any other vertex of $V_{-,-}$ ($V_{+,+}$) between v and the $(X = X^*)$-plane, or*

- *there exists a Y-line with integer coordinates in the $(Z = z)$-plane that intersects v and does not intersect any other vertex of $V_{-,-}$ ($V_{+,+}$) between v and the $(Y = Y^*_-)$-plane ($(Y = Y^*_+)$-plane).*

A vertex is called hidden at level z *if it is not exposed.*

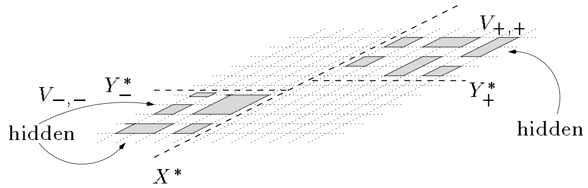

Figure 6: Examples of hidden vertices (we only show the cross-section with one Z-plane). All vertices not marked otherwise are exposed.

Hence, any edge (v, w) between $V_{-,-}$ and $V_{+,+}$ must be routed in a $(Z = z)$-plane such that both v and w are exposed at level z.

The crucial observation is now that if X and Y (the dimensions of the drawing) are small, then not very many vertices are exposed at any one level. This leads to a contradiction, because then not all edges can be routed. More precisely:

Claim: $X + Y \geq 2^{-8}n$.

To prove this claim, assume to the contrary that $X + Y \leq 2^{-8}n$. In particular therefore, at most $2^{-8}n$ vertices of $V_{-,-}$ can be exposed at any one given level, simply because there are only $2^{-8}n$ possible grid lines, each of which can only intersect at most one exposed vertex.

We say that a vertex v is *active* at level z if the $(Z = z)$-plane intersects v, and *inactive* otherwise. Recall that all vertices in $V_{-,-}$ are intersected by the $(Z = Z')$-plane, so they are all active on level Z'. Assume that a vertex $v \in V_{-,-}$ is hidden on level $z - 1 \geq Z'$, but exposed on level z. Hence, some other vertex $w \in V_{-,-}$ was "blocking" v at level $z - 1$. Since w does not block v at level z, w must have disappeared, i.e., w became inactive at level z. Hence, every time one vertex becomes exposed, another vertex must become inactive.

The precise argument is now as follows. Sweep a $(Z = Z_0)$-plane from smaller to larger values of Z, starting at $Z = Z'$. Initially, all vertices in $V_{-,-}$ are active (there are at least $\frac{7}{256}n$ many of them), and at most $2^{-8}n = \frac{1}{256}n$ of them are exposed.

During the sweep, more and more vertices become inactive, and hence more and more vertices become exposed. At some point, an integer $Z^*_{-,+}$ is enountered where for the first time $\geq \frac{3}{256}n$ vertices of $V_{-,-}$ are inactive. Denote these vertices by $V_{-,-,-}$; hence none of them is exposed on any level $z \geq Z^*_{-,+}$.

At most $\frac{1}{256}n$ vertices were exposed on level Z', and at most $\frac{3}{256}n$ vertices became exposed on level $Z' + 1, \ldots, Z^*_{-,+} - 1$, because at most $\frac{3}{256}n$ vertices became inactive on these levels. Hence there are at least $|V_{-,-}| - \frac{1}{256}n - \frac{3}{256}n \geq \frac{3}{256}n$ vertices that are not exposed on any level between Z' and $Z^*_{-,+} - 1$. Denote these vertices as $V_{-,-,+}$.

With a similar argument, we can find an integer $Z^*_{+,+}$ such that at least $\frac{3}{256}n$ vertices of $V_{+,+}$ are inactive on any level $\geq Z^*_{+,+}$ (denote them $V_{+,+,-}$), and at least $\frac{3}{256}n$ vertices of $V_{+,+}$ are not exposed on any level between Z' and $Z^*_{+,+}-1$ (denote them $V_{+,+,+}$). See Figure 7.

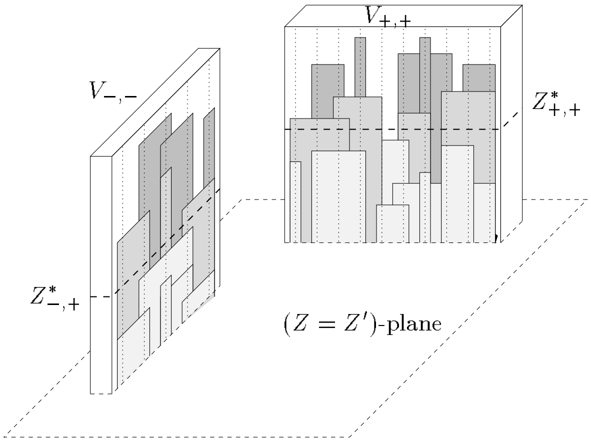

Figure 7: We split the vertices of $V_{-,-}$ such that the vertices in $V_{-,-,+}$ (shown darker) are hidden on all levels between Z' and $Z^*_{-,+}$, and the vertices in $V_{-,-,-}$ (shown lighter) are inactive on all levels $\geq Z^*_{-,+}$. Similarly we split $V_{+,+}$. (We simplify the picture by only showing the projection of the vertices onto the bounding box, and only considering one of the possible directions of exposure.)

Without loss of generality, we may assume that $Z_{-,+} \leq Z_{+,+}$. Therefore, there exists no integral level $z \geq Z'$ on which both a vertex in $V_{-,-,-}$ is active and a vertex in $V_{+,+,+}$ is exposed. Hence none of the edges between these two sets (which exist because we have a Ramanujan-graph) can be routed on a $(Z = z)$-plane with $z \geq Z'$.

Now we repeat the argument for the layers below Z', applied only to vertices in $V_{-,-,-}$ and $V_{+,+,+}$, respectively. Recall that $|V_{-,-,-}| \geq \frac{3}{256}n$. We can find an integer $Z^*_{-,-} < Z'$ such that at least $\frac{1}{256}n$ vertices of $V_{-,-,-}$ are inactive on any level $\leq Z^*_{-,-}$ (call them $V_{-,-,-,+}$), and at least $\frac{3}{256}n - \frac{1}{256}n - \frac{1}{256}n = \frac{1}{256}n$ vertices of $V_{-,-,-}$ (call them $V_{-,-,-,-}$) are hidden on any level between Z' and $Z^*_{-,-}+1$.

We can also find a value $Z^*_{+,-} < Z'$ such that at least $\frac{1}{256}n$ vertices of $V_{+,+,+}$ are inactive on any level $\leq Z^*_{+,-}$ (call them $V_{+,+,+,+}$), and at least $\frac{3}{256}n - \frac{1}{256}n - \frac{1}{256}n = \frac{1}{256}n$ vertices of $V_{+,+,+}$ (call them $V_{+,+,+,-}$) are hidden on any level between Z' and $Z^*_{+,-}+1$.

Without loss of generality, we may assume that $Z^*_{-,-} \leq Z^*_{+,-}$. Therefore, there exists no integral level $z \leq Z'$ on which both a vertex in $V_{-,-,-,-}$ is active and a vertex in $V_{+,+,+,+}$ is exposed. Hence none of the edges between these two sets (which exist because we have a Ramanujan-graph and $d \geq 16(256)^2 = 2^{20}$) can be routed on a $(Z = z)$-plane with $z \leq Z'$. But we also showed before that none of these edges can be routed in a $(Z = z)$-plane with $z \geq Z'$, and not in an X-plane or a Y-plane either. Contradiction, hence the claim follows.

Thus we have now shown that $X + Y \geq 2^{-8}n$. Assume without loss of generality that $X = \max\{X, Y\}$, then $X \geq 2^{-9}n$. There are at least $\frac{1}{2}(\frac{7}{256})^2 \cdot dn$ edges between $V_{-,-}$ and $V_{+,+}$, since each of these sets contains at least $\frac{7}{256}n$ vertices. All these edges intersect the $(X = X^*)$-plane in a point with integer coordinates, therefore $YZ \geq \frac{1}{2}(\frac{7}{256})^2 dn = 49 \cdot 2^{-17}dn$. Combining this with $X \geq 2^{-9}n$, we obtain $XYZ \geq 49 \cdot 2^{-26}dn^2$, which gives the result. □

4 Conclusion and open problems

In this paper, we studied three-dimensional orthogonal 1-bend drawings, and solved two open problems, namely that any 1-bend drawing of K_n has volume $\Omega(n^3)$ and degree-restricted 1-bend cube-drawings are impossible for graphs with $\Omega(\Delta n)$ edges. We also provided lower bounds for 1-bend cube-drawings. The lower bounds can be generalized to any graph for which any cut contains many edges, in particular to Ramanujan-graphs.

We leave a number of open problems:

- Does every graph have a 1-bend drawing of volume $O(\Delta n^2)$? It is easy to construct such a drawing if crossings are allowed, by splitting the edges into $\Delta+1$ matchings, and assigning a separate Z-plane to each matching, similarly as in [BSWW99]. Can a graph be split in $O(\Delta)$ matchings such that for a suitable vertex order all matchings are without crossing?

  If the answer is yes, does every graph have a 1-bend drawing of volume $O(mn)$?

- Does every graph have a 1-bend cube-drawing of volume $O(\Delta^3 n)$? If the answer is yes, does every graph have a 1-bend drawing of volume $O(\Delta^2 m)$?

- What is the correct lower bound for 2-bend drawings? There are drawings of size $O(n^3)$ for K_n [BSWW99] as well as $O(\Delta n^2)$ for all graphs [Bie98]. Is the lower bound $\Omega(\Delta n^2)$, as for the 1-bend case?

- What is the correct lower bound for 3-bend drawings? There are drawings of size $O(n^{2.5})$ for K_n [BSWW99], and this is optimal [BSWW99]. For Ramanujan-graphs, the lower bound is $\Omega(\Delta n^{1.5})$

[BTW00], but it is not known whether every graph has a 3-bend drawing of volume $O(\Delta n^{1.5})$. (Such drawings exist with 4 bends per edge [BTW00]; 3-bend drawings can be constructed with similar techniques if crossings are allowed.)

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