

On Bisectors for Convex Distance Functions in 3-Space

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Abstract

We investigate the structure of the bisector of point sites under arbitrary convex distance functions in three dimensions. Our results show that it is advantageous for analyzing bisectors to consider their central projection on the unit sphere, thereby reducing by one the dimension of the problem. From the concept of “silhouettes” and their intersections we obtain simple characterizations of important structural properties like the number of connected components of the bisector of three sites. Furthermore, we prove that two related bisectors of three sites may intersect in permuted order.

Key words: Bisector, convex distance function, Voronoi diagram, 3D.

1 Introduction

Voronoi diagrams for general distance functions in 3-dimensional space are interesting and have important applications, but not much is really known about their structure and how to compute them. Most of the few known results focus on their complexity. Boissonat et al. [5] show an upper bound of $O(n^2)$ for the complexity of a Voronoi diagram of n point sites under L_1 and L_∞ , as well as for a tetrahedral distance, and generalizations of this for higher dimensions. Tagansky [23] obtains a more general bound of $O(k^3\alpha(k)n^2 \log n)$ for polyhedral distances with k facets in 3-space. Lê [16] shows that the complexity of Voronoi diagrams under L_p distances is bounded in any dimension, independent of p . Chew et al. [7] prove an upper bound of $O(n^2\alpha(n) \log n)$ for the complexity of a Voronoi diagram of lines under a polyhedral distance.

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Two-dimensional Voronoi diagrams based on convex distance functions were first studied by Shamos and Hoey [22] for L_2 , by Lee [19] for the other L_p -metrics, by Widmayer et al. [24] for distance functions defined by convex polygons, and, at the same time, by Chew and Drysdale [6] in the general case. Since convex distance functions are a natural generalization of the Euclidean distance, investigating their Voronoi diagrams is a natural and necessary step towards a unifying theory on Voronoi diagrams, as offered in dimension 2 by the concept of abstract Voronoi diagrams; see Mehlhorn et al. [20] and Klein et al. [12, 13, 14]. For a survey on Voronoi diagrams we refer to Aurenhammer [2] or Aurenhammer and Klein [3], for applications see Chew and Drysdale [6] or the survey paper by Schwartz and Sharir [21]. A further generalization of distances is proposed by Icking et al. [10]; here, every site is associated its own, different distance function.

One of the reasons for the lack of results on Voronoi diagrams for higher dimensions under arbitrary convex distance functions is the surprising, really abnormal, structure of the bisectors which behave totally different from what is known for the Euclidean distance. An example has been presented in [9], where Icking et al. show that the bisector of four sites may consist of arbitrarily many single points, even for a strictly convex and smooth distance and for sites in general position. Their structural results on bisectors for strictly convex distances in two and three dimensions are generalized to smooth distances in arbitrary dimensions by Lê [15]. In [17] he proves that for non-smooth distances in 3-space the bisector of three sites may consist of many disconnected pieces, and in [18] he describes an algorithm which is suitable for ellipsoid distances.

There is an astonishing result by Goodey [8] concerning ellipsoids in any dimension greater or equal to three. Applied to our 3-dimensional case it says that for any convex, non-ellipsoidal body K in \mathbf{R}^3 there are two homothetic copies of K such that the intersection of their boundaries is not planar. So we can always find four non-coplanar points in the intersection of their boundaries, and therefore the bisector of these four sites under distance K contains at least two points, compare [9] or our construction in Section 4. This means that it is extremely hard, or impossible, to find convex distance functions in 3-space which can guarantee to have only one point in the bisector of four sites in general position. In other words, the “surprising” behavior of the 4-bisectors described in [9] is not an exception, but the rule: it strikes nearly any convex distance, except for simplex distances and the affine transforms of the Euclidean distance.

In this paper we prove new results on the behavior of the bisector of three sites. In Section 2 we review some definitions. In Section 3 we investigate the structure of the bisector. The central projection on the unit sphere turns out as a useful means, the behavior of the bisector can be read from the intersection behavior of the silhouettes on the unit sphere. From these results it should be not difficult to derive a sweep-line algorithm for computing the bisector of three sites for e. g. polyhedral distances. In Section 4 we show a surprising result on the intersection behavior of two or more related bisectors of three sites: they may intersect in permuted order. All these results are important steps towards the construction of such Voronoi diagrams.

2 Definitions, assumptions, and simple results

Let C be a compact, convex body in 3-space (not necessarily symmetric, smooth, or strictly convex) which contains the origin O in its interior. For two points a, q , we translate C by vector a and consider the ray $\overrightarrow{a}q$ from a through q . Let v denote the unique point on the boundary of C hit by this ray; see Figure 1. Then by

$$d(a, q) = \frac{\|q - a\|}{\|v - a\|}$$

a *convex distance function* d is defined. Here $\|q - a\|$ denotes the Euclidean distance between q and a . Clearly, C is the *unit ball* of all points q satisfying $d(0, q) \leq 1$, equality holding only for the points on the boundary of C , the *unit sphere* ∂C . Well-known examples of convex distance functions are the L_p -metrics, $1 \leq p \leq \infty$, defined by $\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + |x_3|^p}$, among them the Euclidean distance, L_2 .

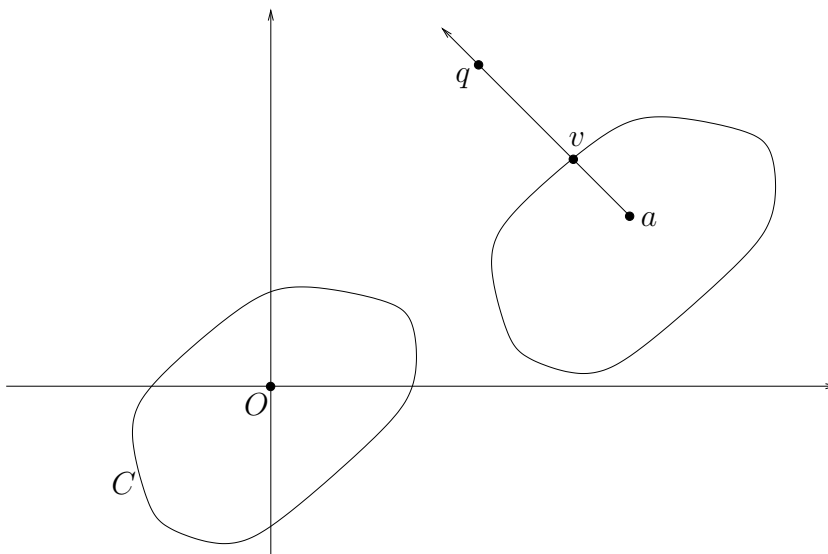


Figure 1: A convex distance function.

The *bisector* $B(a_1, a_2)$ of two sites, a_1 and a_2 , is the set $\{q; d(a_1, q) = d(a_2, q)\}$ of all points whose distance from a_1 equals the distance from a_2 . For brevity, we write $B(a_1, a_2, a_3)$ instead of $B(a_1, a_2) \cap B(a_1, a_3)$ and $B(a_1, a_2, a_3, a_4)$ instead of $B(a_1, a_2, a_3) \cap B(a_1, a_4)$ for the bisector of three resp. four points. We also speak of 2-bisectors, 3-bisectors, and 4-bisectors, if only the number of participating sites is meant.

In some degenerate cases a 2-bisector can contain 3-dimensional pieces, branchings, or self-intersections. To avoid this, see Lemma 2, we make the following *assumption on general position*: no line through two sites is parallel to a line segment which is contained in ∂C . This assumption is appropriate because a non-general position does not persist after a small perturbation of the sites.

Given n point sites, a_1, \dots, a_n , the *Voronoi diagram* based on a convex distance function d can be defined in the usual way. With each site a_i , the *Voronoi region* containing all points q satisfying $d(a_i, q) = \min_{1 \leq j \leq n} d(a_j, q)$ is associated. The boundary of the region of a_i consists of pieces of bisectors $B(a_i, a_j)$ where $i \neq j$.

Let Γ_{ij} be the set of all points on the surface of C that admit a tangent parallel to $a_i a_j$. We call Γ_{ij} the *silhouette* for direction $a_i a_j$. For example, the silhouette on a polytope consists only of edges of the polytope, due to general position.

Under the assumption of general position we obtain a number of useful properties.

Lemma 1 *The silhouette Γ_{12} for direction $a_1 a_2$ is a simple closed curve.*

Proof. The silhouette is (doubly) connected due to the convexity of C . It is a simple curve due to the assumption of general position. \square

By Lemma 1, a silhouette cuts the surface of C into two open “half-spheres” which are homeomorphic to a plane. Let H_{ij} be the relatively open half-sphere of C bounded by the silhouette Γ_{ij} that intersects the ray $\overrightarrow{O(a_j - a_i)}$. The two half-spheres H_{ij} and H_{ji} share the same boundary Γ_{ij} , and they represent a disjoint partition of ∂C , i. e., $H_{ij} \dot{\cup} \Gamma_{ij} \dot{\cup} H_{ji} = \partial C$.

Any point $p \in \mathbf{R}^3 \setminus \{a_i\}$ can be mapped to the intersection point of the ray $\overrightarrow{a_i p}$ and $\partial C + a_i$, this is called the *central projection* centered at a_i .

Lemma 2 *The bisector $B(a_1, a_2)$ of two sites is homeomorphic to a plane.*

Proof. The central projection centered at a_1 is a homeomorphism of $B(a_1, a_2)$ onto $H_{12} + a_1$, which is homeomorphic to a plane. More details of the proof can be found in [9] where the claim is proven under the assumption of strict convexity of C , but the proof still holds, without any modification, under our weaker assumption of general position. \square

Remark that the assumption on general position is crucial, since the converse of Lemma 2 is also true: in non-general position, i. e., if the line $a_1 a_2$ is parallel to a line segment of ∂C , the bisector $B(a_1, a_2)$ is definitely not homeomorphic to a plane. As examples one may look at the bisector for a cylindrical unit ball, which contains an unbounded 3-dimensional component, or the bisector for a cone, which consists of an infinite triangle glued to a 2-dimensional surface, in such a special position of the two sites.

3 The structure of the bisector of three sites

There is a close relationship between the bisector of three sites in 2-space and in 3-space, as described in [9] and [17]. Let p be a point of $B(a_1, a_2, a_3)$ in 3-space, and let v_i be its central projection centered at a_i , for $i = 1, 2, 3$, see Figure 2. The plane π through v_1, v_2, v_3 is parallel to the plane through a_1, a_2, a_3 . Let K be the intersection of π and $C + a_1$. We choose an interior point w_1 in K , and let $w_i = w_1 - a_1 + a_i$, for $i = 2, 3$. The lines $w_i v_i$, $i = 1, 2, 3$, intersect in a common point r that is the bisector of w_1, w_2, w_3 with respect to the unit circle K in the plane π .

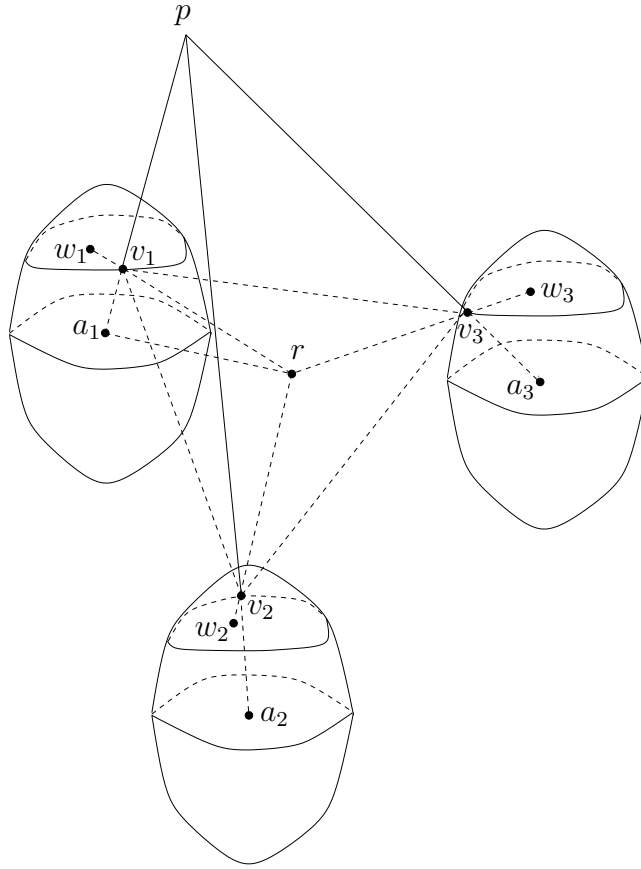


Figure 2: The construction of a bisector point p by central projection shows the close relationship between the 3-bisectors in two and three dimensions.

Conversely, for each plane π parallel to a_1, a_2, a_3 that intersects $C + a_1$ we consider a 2-dimensional bisector problem using $(C + a_1) \cap \pi$ as the unit circle. It is not hard to see that we can construct the corresponding bisector point r on π , if it exists, and obtain the points v_i on $(C + a_i) \cap \pi$. From this, we finally get a point $p \in B(a_1, a_2, a_3)$ as the intersection of the lines $a_i v_i$.

The mapping from p to the plane through the points v_i , i. e. the *construction by central projection*, is continuous in both directions, see [11] and [9, 17].

For the intersection of half-spheres we introduce an abbreviation, let $H_{ijk} = H_{ij} \cap H_{ik}$. This notation is, of course, commutative in the second and third index, i. e. $H_{ijk} = H_{ikj}$.

Lemma 3 *The intersected half-spheres $H_{123}, H_{213},$ and H_{312} are disjoint and partition the unit sphere, i. e., for their closures we have $\overline{H_{123}} \cup \overline{H_{213}} \cup \overline{H_{312}} = \partial C$.*

Proof. We have $H_{123} \cap H_{213} = \emptyset$ by definition, due to $H_{12} \cap H_{21} = \emptyset$.

The silhouettes $\Gamma_{12}, \Gamma_{13},$ and Γ_{23} have at least two points in common, namely the points of ∂C touched by the supporting planes parallel to a_1, a_2, a_3 from above and below. At these points, each silhouette separates into two branches. The boundary of H_{123} consists of one branch of Γ_{12} and one branch of Γ_{13} . The “unused” branches are contained in $\overline{H_{312}}$ and $\overline{H_{213}}$, respectively. Therefore, they must partition the unit sphere. \square

In some cases, the intersected half-spheres H_{ijk} can be empty or disconnected. The 3-bisectors copy their behavior, as the next two lemmas show.

Lemma 4 *The bisector $B(a_1, a_2, a_3)$ is not empty iff all three intersected half-spheres H_{123} , H_{213} , and H_{312} are not empty.*

Proof. Let p be a point of $B(a_1, a_2, a_3)$. Its central projection centered at a_1 lies in $(H_{12} \cap H_{13}) + a_1 = H_{123} + a_1$, compare the proof of Lemma 2. So $H_{123} \neq \emptyset$, and analogously for the other intersected half-spheres.

Conversely, assume that H_{123} , H_{213} , and H_{312} are all not empty. We consider a plane π parallel to a_1, a_2, a_3 which intersects the unit sphere in more than just one point. For brevity, we write $H'_{12} = H_{12} \cap \pi$, etc., for the intersection of the half-spheres with the plane. It is clear that not all three of H'_{123} , H'_{213} , and H'_{312} can be empty, by Lemma 3.

We even show that at most one of them is empty. So assume the contrary, say $H'_{123} = H'_{213} = \emptyset$. Then $H'_{12} \cap H'_{13} = \emptyset$, thus $H'_{12} = H'_{31}$, and analogously $H'_{21} = H'_{32}$. Therefore, $H'_{312} = H'_{31} \cap H'_{32} = H'_{12} \cap H'_{21} = \emptyset$, a contradiction.

Now we consider all possible positions of the plane π . Due to the relative openness of H_{123} , H_{213} , and H_{312} and the fact just proven, there must be a position of π such that all three of H'_{123} , H'_{213} , and H'_{312} are non-empty. More precisely, there must be such a position in any connected component of H_{123} .

For this particular position of π , we consider a 2-dimensional bisector problem using the unit circle $C \cap \pi$. It is easy to see that we obtain a point of the bisector $B(a_1, a_2, a_3)$ by using the construction by central projection presented at the beginning of this section. \square

The bisector of three sites can be disconnected and each component is homeomorphic to a line, as already observed in [17]. The reasons for this become clear in the next lemma.

Lemma 5 *The bisector $B(a_1, a_2, a_3)$ is connected iff all three intersected half-spheres H_{123} , H_{213} , and H_{312} are connected. The number of connected components of $B(a_1, a_2, a_3)$ plus 2 equals the number of connected components of the three sets.*

Proof. Assume that $B(a_1, a_2, a_3)$ is not empty and connected. From Lemma 4 we know that the three sets H_{123} , H_{213} , and H_{312} are not empty, and in its proof we have even seen that we can find, using the construction by central projection, the image of a bisector point in each connected component of $H_{123} + a_1$, etc. But the central projection is continuous and therefore maps connected sets to connected sets, so H_{123} etc. must be connected. For an empty $B(a_1, a_2, a_3)$, which is connected by definition, one of the three sets must be empty by Lemma 4, and therefore the other two must be connected.

Conversely, if H_{123} , H_{213} , and H_{312} are all connected, then the construction by central projection delivers one bisector point for every plane parallel to a_1, a_2, a_3 that intersects $H_{123} + a_1$, $H_{213} + a_2$, and $H_{312} + a_3$. Since this construction is a continuous mapping of a connected set of planes to the bisector, $B(a_1, a_2, a_3)$ must be connected, too.

For the number of connected components we consider a moving plane parallel to a_1, a_2, a_3 , sweeping the whole unit sphere, and we observe the bisector points constructed by central projection. For the first part of $B(a_1, a_2, a_3)$ that is constructed, we “use up” one connected component of each of H_{123}, H_{213} , and H_{312} . Whenever a new piece of $B(a_1, a_2, a_3)$ begins, this is caused by a new connected component of one of the three sets, because any connected component of one of them makes a non-empty contribution to the bisector, see the proof of Lemma 4. \square

We give a simple example for constructing a disconnected 3-bisector. Figure 3 shows three sites a_1, a_2, a_3 as well as the intersections of the unit spheres centered at these sites with a plane parallel to a_1, a_2, a_3 , at three levels, see Figure 4 for an impression of the unit sphere in 3-space. At the three levels considered, the intersection is a triangle, but the triangle in the middle is rotated against the triangles above and below. Considering the three situations as planar bisector problems, there is a 3-bisector in the upper and lower case, but no 3-bisector point in the middle. This corresponds to the fact that there is an empty set H'_{213} in the middle position, while all three sets exist in the other situations. Therefore, the 3-bisector in 3-space is interrupted, by Lemma 5.

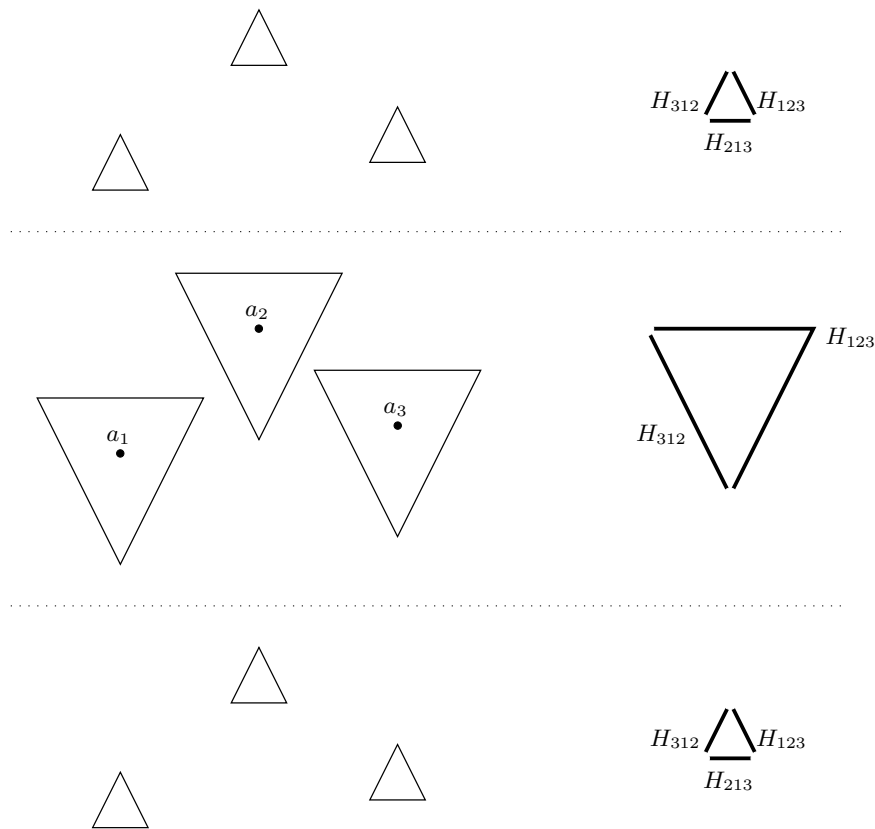


Figure 3: To the left, three planar intersections through three unit spheres as the one shown in Figure 4. To the right, the corresponding sets H_{123}, H_{213} , and H_{312} .

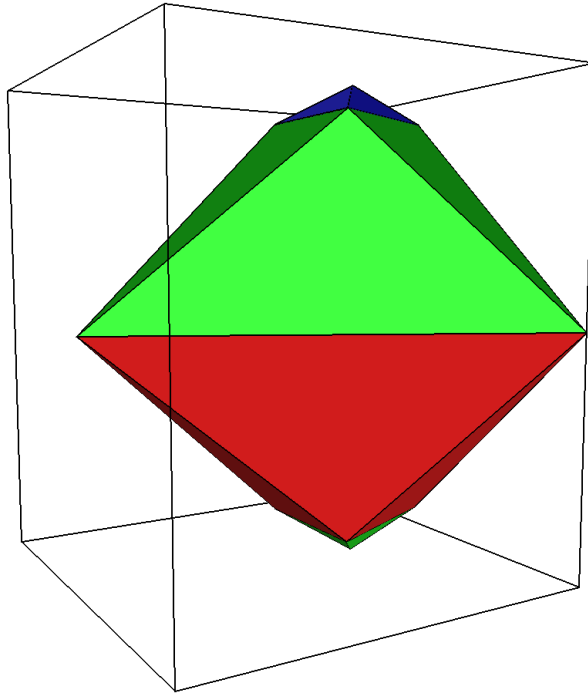


Figure 4: The unit sphere of the example in Figure 3, together with its bounding box.

4 The intersection of two related bisectors of three sites

Now let us consider the bisector $B(a_1, a_2, a_3, a_4)$ of four sites. In the Euclidean case, the bisector of four non-coplanar points is always one point, but for general distances this is much more complicated, see [9]. It may contain curves in \mathbf{R}^3 and can consist of arbitrarily many discrete points or connected components, even for sites in general position, and even if we additionally assume the unit sphere C to be smooth and strictly convex. Here, we reveal another, also deterrent property of $B(a_1, a_2, a_3, a_4)$.

The Voronoi region of a site is bounded by pieces of bisectors, and for computing the structure of such a region one must look closely at the intersection of all bisectors related to this site. From the construction by central projection, see Figure 2, the bisector of three sites, which is always one-dimensional due to general position, is naturally ordered. One might hope that two such bisectors, which appear in the common boundary of two Voronoi regions, intersect nicely, i. e. the intersections appear in the same order (or reverse) on both bisectors. This will be disproved in the following.

The intersection of two related 3-bisectors, say $B(a_1, a_2, a_3)$ and $B(a_1, a_2, a_4)$, is the 4-bisector $B(a_1, a_2, a_3, a_4)$. We consider a point, p , of the 4-bisector and its central projections, v_i , centered at a_i , for $i = 1, 2, 3, 4$; compare Figure 2. The tetrahedron $T(v_1, v_2, v_3, v_4)$ is homothetic to the tetrahedron $T(a_1, a_2, a_3, a_4)$ and also homothetic to $T(v_1 - a_1, v_2 - a_2, v_3 - a_3, v_4 - a_4)$. In this way, for each point of the 4-bisector we have one tetrahedron homothetic to $T(a_1, a_2, a_3, a_4)$ whose vertices lie on ∂C .

Now we imagine several of such homothetic tetrahedra whose vertices lie on ∂C , and we consider a sweep plane parallel to a_1, a_2, a_3 , passing through C . It is also parallel to one of the faces of the tetrahedra. The plane visits these faces in the

same, natural order as the corresponding points of the 4-bisector lie on $B(a_1, a_2, a_3)$. This holds because the central projection of the 3-bisector $B(a_1, a_2, a_3)$ on $\partial C + a_1$ is strictly monotonic in the direction orthogonal to the plane through a_1, a_2, a_3 .

But this order in which the tetrahedra are visited is not necessarily the same for all four faces. We can indeed construct an example of a 4-bisector containing at least seven points such that the corresponding tetrahedra are visited in totally different order, depending on the face.

To this end, we define seven tetrahedra as shown in the left picture of Figure 5. Their coordinates are shown in Table 1, remark that all 28 vertices are in convex position.

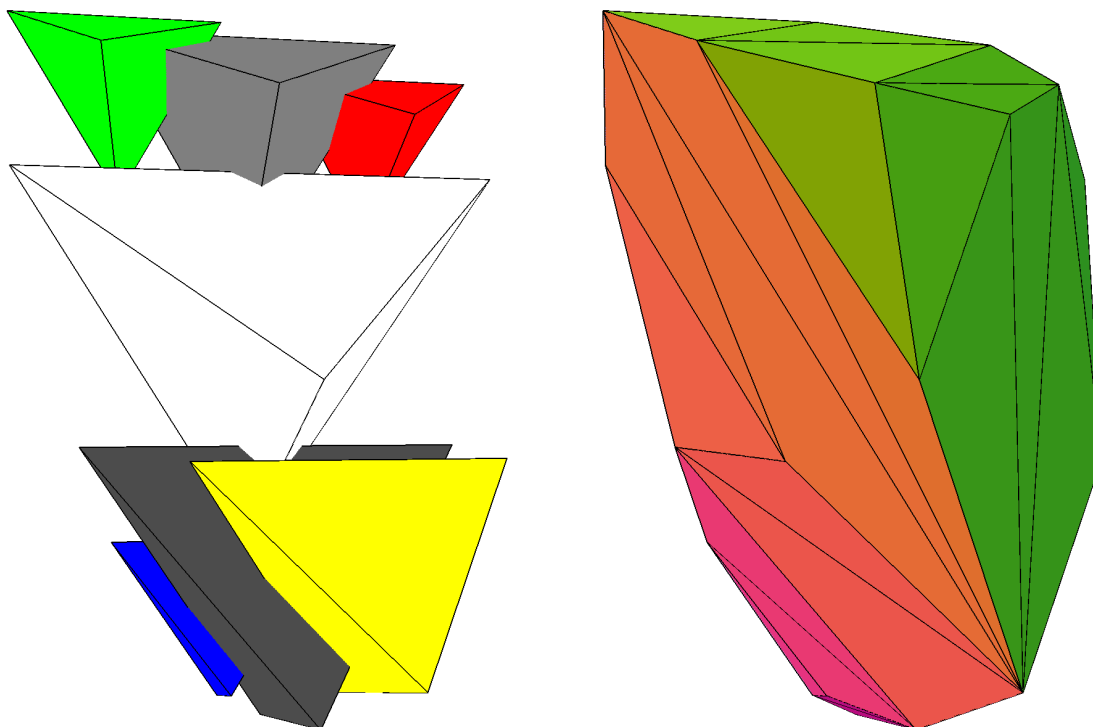


Figure 5: The left picture shows seven homothetic tetrahedra whose parallel faces appear in permuted order. The right picture shows their convex hull, we observe that all vertices of the tetrahedra are in convex position, i. e. they appear as vertices of the convex hull. The convex hulls here and in Figure 4 were computed by Quickhull [4], the pictures were rendered using Geomview [1].

The four families of parallel faces of the seven tetrahedra are visited in the orders given by the following four rows.

1	2	3	4	5	6	7
5	6	7	4	3	2	1
7	6	5	4	1	2	3
3	2	1	4	7	6	5

So, if we now choose the convex hull of the seven tetrahedra, see the right picture in Figure 5, or any other convex body C containing the 28 vertices in ∂C as our

T_1 :	-14.0	18.0	2.0	T_5 :	2.0	-18.0	14.0
	-2.0	18.0	2.0		2.0	-18.0	2.0
	-8.0	10.0	8.0		8.0	-10.0	8.0
	-8.0	10.0	-4.0		-4.0	-10.0	8.0
T_2 :	-9.0	19.0	0.0	T_6 :	-9.0	-7.0	0.0
	9.0	19.0	0.0		9.0	-7.0	0.0
	0.0	7.0	9.0		0.0	-19.0	9.0
	0.0	7.0	-9.0		0.0	-19.0	-9.0
T_3 :	2.0	18.0	-2.0	T_7 :	-2.0	-18.0	-2.0
	14.0	18.0	-2.0		-2.0	-18.0	-14.0
	8.0	10.0	4.0		4.0	-10.0	-8.0
	8.0	10.0	-8.0		-8.0	-10.0	-8.0
T_4 :	-13.5	9.0	0.0				
	13.5	9.0	0.0				
	0.0	-9.0	13.5				
	0.0	-9.0	-13.5				

Table 1: The coordinates of the seven homothetic tetrahedra of Figure 5.

unit sphere, and if we choose four sites that lie on the vertices of a tetrahedron which is homothetic to the other seven, then the given permutations also apply to the order in which the pieces of the 4-bisector appear on the four 3-bisectors. This strange behavior is illustrated in Figure 6, which schematically shows how the central projections centered at a_1 of the three 3-bisectors $B(a_1, a_2, a_3)$, $B(a_1, a_2, a_4)$, and $B(a_1, a_3, a_4)$ and their intersections look like.

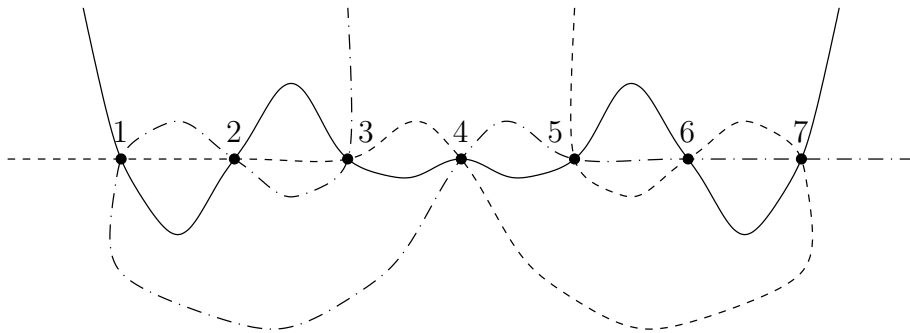


Figure 6: Schematic view on the intersections of the three related 3-bisectors under the polyhedral distance of Figure 5.

The described phenomenon does, of course, does not depend on the fact that the unit sphere is a polyhedron. We can easily construct a strictly convex and smooth body whose surface also passes through the 28 vertices of the seven tetrahedra.

5 Conclusions

The nice relationships between the behavior of the 3-bisectors and the intersections of the silhouettes shows that there should be a rather simple sweep-line algorithm to compute 3-bisectors under arbitrary convex distance functions. For computing whole Voronoi regions, however, one has to be prepared to meet strange situations like the one described in Section 4, which can not be “defined away” by assumptions on general position of the sites or on smoothness or strict convexity of the unit sphere.

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