Ramsey-type results for unions of comparability graphs and convex sets in restricted position

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Abstract

Given a graph on n vertices which is the union of two comparability graphs on the same vertex set, it always contains a clique or independent set of size $n^{\frac{1}{3}}$. On the other hand, there exist such graphs for which the largest clique and independent set are of size at most $n^{0.4118}$. Similar results are obtained for graphs which are a union of a fixed number k of comparability graphs. We also show that the same bounds hold for unions of perfect graphs. A geometric application is included.

1 Introduction

Let $P = (V, \prec)$ be a partially ordered set on $V = \{v_1, v_2, \ldots, v_n\}$. The *comparability graph* of P, $G = (V, E)$ is a graph such that $(v_i, v_j) \in E$ if and only if either $v_i \prec v_j$ or $v_j \prec v_i$.

Given a graph $G = (V, E)$, let $\omega(G)$ be its clique number, namely the size of a largest clique in G and $\alpha(G)$ be its independence number, namely the size of a largest independent set in G. Write $\chi(G)$ for the chromatic number of G. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G, including G itself (see [B98]).

For $k \geq 1$, put $\mathcal{C}_k$ = the class of graphs which are unions of k comparability graphs on the same vertex set. Write $f_k(n)$ for the minimum of $\max(\omega(G), \alpha(G))$, over all graphs $G \in \mathcal{C}_k$ on n vertices. Define $\mathcal{P}_k, g_k(n)$ analogously for perfect graphs (i.e. $\mathcal{P}_k$ is the class of graphs which are unions of k perfect graphs). For $k \geq 2$, in Section 2 we obtain

Theorem 1.1

$$n^{\frac{1}{k+1}} \leq g_k(n) \leq f_k(n) \leq n^{\frac{1+\log k}{k}}$$

In the case $k = 2$, in Section 3 we prove

Theorem 1.2

$$n^{\frac{1}{3}} \leq g_2(n) \leq f_2(n) \leq n^{0.4118}$$

Let $h(n)$ denote the maximum number with the property that given a family of n convex sets in the plane, one can always choose $h(n)$ of them which are either pairwise disjoint or pairwise intersecting. It was proved [LMPT94], [KPT97]

$$n^{\frac{1}{5}} \leq h(n) \leq n^{\log_{27} 4} \leq n^{0.4207}$$

In [LMPT94] four partial order relations are defined, such that any two disjoint convex sets in the plane are comparable using at least one of them (see Figure 1). Let $\pi(C)$ denote the projection of C on the x-axis. For $A \cap B = \emptyset$

- $A <_1 B$ if $\pi(A) \subseteq \pi(B)$ and A lies below B ("below" means in the y-axis direction).
- $A <_2 B$ if $\pi(A) \subseteq \pi(B)$ and A lies above B.
- $A <_3 B$ if the left endpoint of $\pi(A)$ is to the left of the left endpoint of $\pi(B)$, the right endpoint of $\pi(A)$ is to the left of the right endpoint of $\pi(B)$, and if $\pi(A)$ and $\pi(B)$ overlap, A lies below B in that part.
- $A <_4 B$ if the left endpoint of $\pi(A)$ is to the left of the left endpoint of $\pi(B)$, the right endpoint of $\pi(A)$ is to the left of the right endpoint of $\pi(B)$, and if $\pi(A)$ and $\pi(B)$ overlap, A lies above B in that part.

We say that a family of convex sets is in *restricted position of type* (i, j), where $1 \leq i < j \leq 4$, if any two disjoint sets in the family are comparable by $<_i$ or $<_j$. For example, a family of convex sets is in restricted position of type $(1, 2)$ when for any two disjoint convex sets their intervals of projection on the x-axis are comparable by inclusion.

Let $h_{(i,j)}(n)$ denote the maximum number with the property that given a family of n convex sets in restricted position of type (i, j) in the plane, one can always choose $h_{(i,j)}(n)$ of them which are either pairwise disjoint or pairwise intersecting. In Section 4, we show

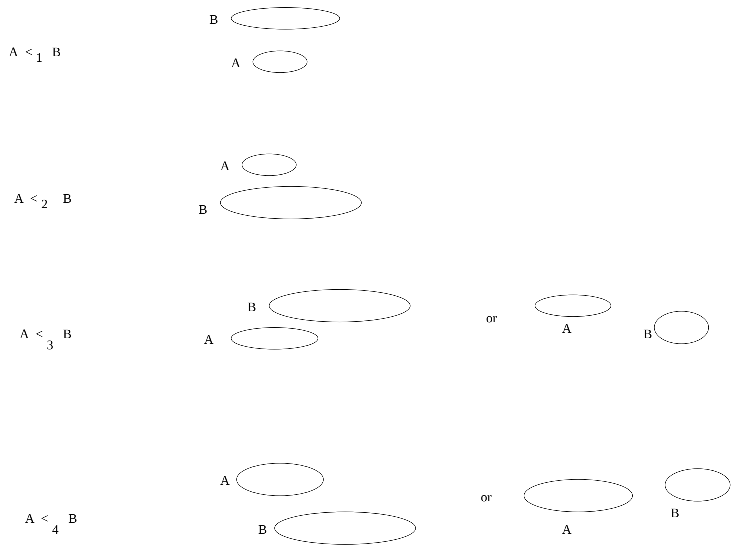

Figure 1: The order relations $<_1, <_2, <_3, <_4$

Theorem 1.3 *For* $1 \leq i < j \leq 4$,

$$n^{\frac{1}{3}} \leq h_{(i,j)}(n) \leq n^{\log_{27} 4} \leq n^{0.4207}$$

Unions of comparability graphs have been used recently in combinatorial geometry, for proving Ramsey-type theorems (see [KPT97], [LMPT94], [PA95], [PT94], [PT99], [TV98], [T99]). In Section 5 we review these related geometric results which motivated our paper.

2 Proof of Theorem 1.1

All comparability graphs are perfect (see [B98]), therefore $g_k(n) \leq f_k(n)$.

We prove the lower bound in the Theorem by induction on k. For $k = 1$, the statement is a direct consequence of the definition of perfect graphs. Suppose that we have already proved the statement for $k - 1$ and for all n. Let $G_i = (V, E_i)$ $(1 \leq i \leq k)$ be perfect graphs, where $|V| = n$. Assume for simplicity that $n = m^{k+1}$ for some integer m. Suppose that the size of any clique in $G(V, E = \cup_{i=1}^{n} E_i)$, is less than $m = n^{\frac{1}{k+1}}$, else the conclusion follows. Since $G_1 = (V, E_1)$ is perfect and $\omega(G_1) < m$, it can be colored by less than m colors, so it contains an independent set V' of size (at least) $n/m = m^k$.

For $i = 2, \ldots, k$, let $G_i' = (V', E_i') = G_i[V']$ be the V'-induced subgraph of G_i. The graphs G_i are perfect, hence G_i' are also perfect. By the induction hypothesis and the assumption, the graph $G' = (V', E' = \cup_{i=2}^{k} E_i')$ contains an independent set I of size m. Then I is independent in G, proving the lower bound. We continue with the proof of the upper bound.

Definition. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. The *ordered product* $G(V, E) = G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$, and edge set

$$E(G) = \{((x_1, x_2), (y_1, y_2)) \mid x_1, y_1 \in V_1, x_2, y_2 \in V_2,$$

$$(x_1, y_1) \in E_1, \text{ or } x_1 = y_1 \text{ and } (x_2, y_2) \in E_2\}$$

Let $k \geq 2$ be a fixed positive integer. By known bounds for Ramsey numbers [AS92] we know that there is a graph $H = (V, E)$ such that $N = |V| = 2^k$, and $\omega(H), \alpha(H) < 2k$. First we show that H is the union of k comparability graphs. Obviously, in H, the largest connected component has at most $|V(H)| = 2^k$ vertices. There exists a bipartite graph $B_1 \subset H$ on V, such that in $H \setminus B_1$ the size of the largest connected component is at most 2^{k-1} (take any balanced bipartition of V). Similarly, we can take a bipartite graph $B_2 \subset H \setminus B_1$, such that in $H \setminus B_1 \setminus B_2$ the size of the largest connected component is at most 2^{k-2}. After k analogous steps, the largest connected component in the remaining graph has just one vertex, so we can cover all edges of H by at most k bipartite graphs on V. Since bipartite graphs are comparability graphs, H is the union of k comparability graphs. Let

$$G = \overbrace{H \times \cdots \times H}^{i \text{ times}}$$

It is easy to see that G is also a union of k comparability graphs. $n = |V(G)| = N^i = 2^{ki}$ and

$$\omega(G), \alpha(G) < (2k)^i = n^{\frac{1+\log k}{k}}$$

This concludes the proof of Theorem 1.1.

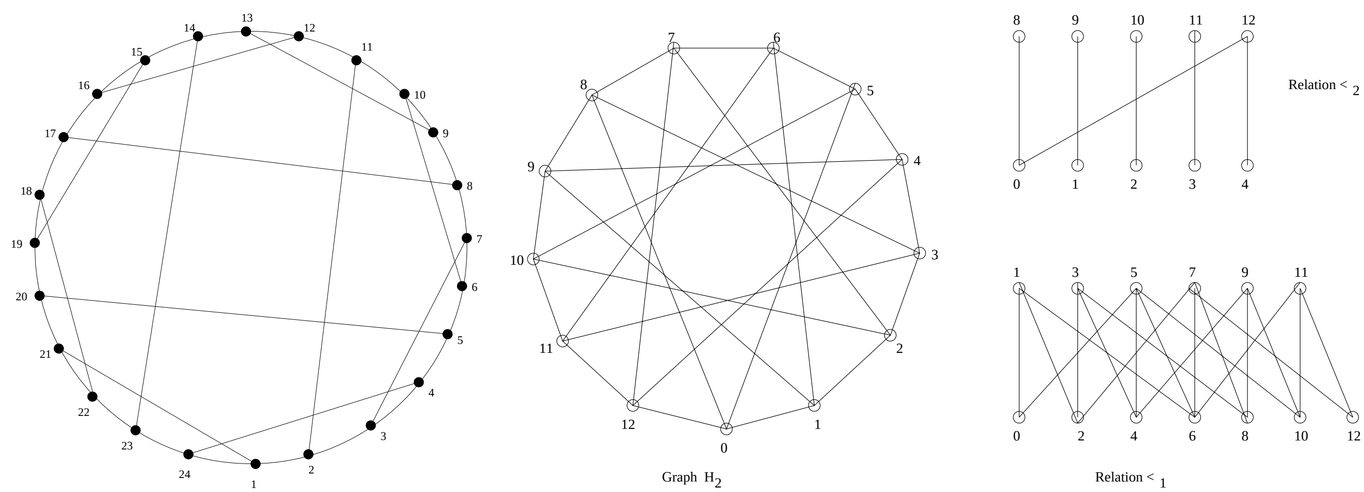

Figure 2: A configuration of 12 segments defining a polygon-disjointness graph H_1 (left), and a decomposition of the graph H_2

3 Proof of Theorem 1.2

We only have to show the upper bound in the Theorem. We first describe a class of graphs which are unions of two comparability graphs. A *polygon-disjointness graph* is a graph whose vertices form a family of convex polygons for which the union of their vertices are in convex position. Each convex polygon gives a vertex in the graph. Two vertices of the graph are joined by an edge when their corresponding polygons are disjoint.

Lemma 3.1 *Any polygon-disjointness graph is in $\mathcal{C}_2$.*

Proof. Omitted in this abstract.

□

Let H_1 be the polygon-disjointness graph on 12 vertices corresponding to the configuration of 12 segments in Figure 2. It has $\omega(H_1) = 4$ and $\alpha(H_1) = 2$. Let H_2 be the graph on 13 vertices drawn in Figure 2 (this graph is used to show that the Ramsey number $R(3, 5) \geq 14$). We know that $\omega(H_2) = 2$ and $\alpha(H_2) = 4$. In Figure 2 a decomposition of H_2 as a union of two comparability graphs is also shown. Put $H = H_1 \times H_2 \in \mathcal{C}_2$. We have $|V(H)| = 156$, $\omega(H) = \omega_1\omega_2 = \alpha_1\alpha_2 = \alpha(H) = 8$. Put

$$G = \overbrace{H \times \cdots \times H}^{i \text{ times}}$$

Then G is still the union of two comparability graphs, $n = |V(G)| = 156^i$, and

$$\omega(G), \alpha(G) = 8^i = n^{\log_{156} 8} < n^{0.4118}$$

This concludes the proof of the Theorem.

4 Proof of Theorem 1.3

The proof of the lower bound in Theorem 1.3 is as in [LMPT94], a repeated application of Dilworth's Theorem [D50] (or it follows directly from Theorem 1.1). We continue with the proof of the upper bound. Consider families of segments having disjoint endpoints and let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be such a system of n segments. We say that $\mathcal{S}$ can be flattened if for every $\epsilon > 0$ there are two discs of radius ϵ at unit distance from each other, and there is another family of n segments $\mathcal{S}' = \{s'_1, \ldots, s'_n\}$ such that s'_i and s'_j are disjoint if and only if s_i and s_j are disjoint, and the endpoints of each s'_i lie one in each disc.

Lemma 4.1 *For $1 \leq i < j \leq 4$, any system $\mathcal{S}$ of segments whose endpoints form the vertex set of a convex polygon, can be flattened such that the resulting system of segments is in restricted position of type (i, j).*

Proof. Omitted in this abstract.

□

In the family $\mathcal{S}_1$ of 27 segments in Figure 3, taken from [KPT97], the largest pairwise disjoint or pairwise intersecting subfamily has size 4. This gives a recursive construction of a family of 27^k segments where the largest pairwise disjoint or pairwise intersecting subfamily has size 4^k; the recursive step replaces a segment of $\mathcal{S}_1$ with a suitable flattened copy of $\mathcal{S}_1$. For $1 \leq i < j \leq 4$, it can be seen that the restricted position of type (i, j) can be maintained at each step using Lemma 4.1. This completes the proof of Theorem 1.3.

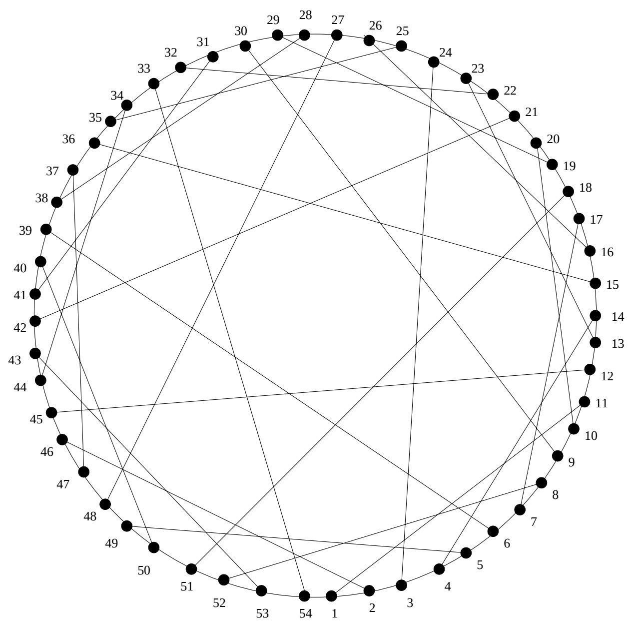

Figure 3: A system of 27 segments

5 Remarks

Probably the first Ramsey-type result that used unions of comparability graphs appears in [LMPT94]; it is shown that among any n convex sets in the plane, there are either $n^{1/5}$ pairwise intersecting or pairwise disjoint. Most likely this bound is not the best possible. The best known result from the other direction is : for infinitely many n, there exists a collection of n convex sets in the plane such that less than $n^{0.4207}$ of them are pairwise intersecting or pairwise disjoint [KPT97].

For some special classes of convex sets there are even stronger results. Any collection of n axis-parallel rectangles contains either $\sqrt{n/(2\log n)}$ pairwise intersecting or pairwise disjoint [LMPT94]. Among n homothetic copies of a convex set in the plane there are either $c\sqrt{n}$ pairwise intersecting or pairwise disjoint. It is the consequence of the following more general result. We say that a convex planar set K-*fat*, if the ratio of the radii of the smallest covering disc R and the largest inscribed disk r satisfies $R/r \leq K$. For any $K > 0$, any collection of n K-fat convex sets in the plane contains $c(K)\sqrt{n}$ pairwise intersecting or pairwise disjoint [P80].

Given a system of n convex polygons for which the union of their vertices are in convex position, one can always choose $c\sqrt{n/\log n}$ of them which are pairwise disjoint or pairwise intersecting. In particular it is true for a system of n segments having their endpoints in convex position. This is a direct consequence of a theorem in [KK97].

A *geometric graph* is a graph drawn in the plane so that the vertices are represented by points in general position and the edges are represented by straight line segments connecting the corresponding points. Using comparability graphs, Pach and Törőcsik obtained the following result [PT94]: any geometric graph of n vertices and k^4n+1 edges contains $k+1$ pairwise disjoint edges. Using similar methods, this bound was improved in [TV98] and recently further improved in [T99]: any geometric graph of n vertices and 2^9k^2n edges contains $k+1$ pairwise disjoint edges.

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