Dissections, Cuts, and Triangulations

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Abstract

We consider two problems on dissections of polygons. In the first problem we consider the minimum number of pieces in dissecting with rectilinear glass cuts an $\frac{m}{n} \times \frac{m}{n}$ rectangle into a unit square. A rectangle is called semi-integer if either its base or its height is an integer. In the second problem we show that no triangulation of a regular polygon can be a dissection of another regular polygon of the same area.

1 Introduction

Dissections of geometric objects have been studied since ancient times [3]. A lot of activity was sparked by Hilbert's address to the 1900 International Congress of Mathematicians, Hilbert conjectured the impossibility of proving merely by dissections the equality of the volume of two tetrahedra with identical basis and equal height, a problem whose solution as described by Euclid uses approximation techniques (See Hilbert's 3rd problem, [5]). The solution suggested by Dehn in [2, 1] is based on the notion of "invariance" of a polyhedron, which in turn depends on the average weights of the edges and dihedral angles of the polyhedron [7].

In this paper we consider two problems on dissections of polygons. In the first problem we study the minimum number of pieces in dissecting with rectilinear glass cuts an $\frac{m}{n} \times \frac{m}{n}$ rectangle into a unit square, $m > n$. The dissection algorithm consists of $O(\log m)$ iterations of dissections. It dissects the rectangle into

$$2\sum_{i=0}^{k} \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor + O(\log m),$$

rectangular pieces, where $r_0 = m > r_1 = n > \cdots > r_{k+1} = \gcd(m, n)$ is the sequence of integers produced by the computation of $\gcd(m, n)$ using the Euclidean algorithm.

In the second problem we prove an impossibility result, namely we show that for m and n sufficiently large no triangulation of a regular m-gon can be a dissection of another regular n-gon of the same area.

2 Rectilinear Dissections of Rectangles

In this section we consider the following problem concerning rectilinear dissections of rectangles.

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Problem 2.1 *Find a rectilinear dissection of a rectangle having area 1 to a unit square using the minimum number of pieces.*

If the dimensions of the rectangle are $a \times b$ (with $a > b$) then $a \cdot b = 1$. If either a or b is irrational then the problem has no solution. (E.g., the $\sqrt{2} \times 1/\sqrt{2}$ rectangle cannot be dissected to a unit square. See [7].) If the dissections are not necessarily rectilinear then Montucla's dissection (see [3]) will dissect the rectangle into a unit square using at most $\lceil a/b \rceil + 2$ pieces (this dissection is valid regardless of whether or not a, b are rationals).

Therefore we consider only the case where both a and b are rationals. Let $a = m/n$ and $b = n/m$, where m, n are integers and $m > n$. We consider the problem of dissecting an $\frac{m}{n} \times \frac{n}{m}$ rectangle into a unit square using only rectilinear glass cuts. It is easy to see that by merely scaling the problem is equivalent to dissecting a rectangle of dimensions $m^2 \times n^2$ into the $mn \times mn$ square. Let $p(m, n)$ be the minimum number of pieces in dissecting the $m^2 \times n^2$ rectangle into the $mn \times mn$ square. In the sequel we prove th following lemma.

Lemma 2.1 *If* $m > n$ *then*

$$p(m, n) \leq 2 \cdot \left\lfloor \frac{m}{n} \right\rfloor + p(n, m \bmod n).$$

Proof We start with a rectangle R of dimensions $m^2 \times n^2$. The dissection is in two steps.

In the first step we dissect the original rectangle R with vertical glass cuts (see Figure 1). Each piece is a rectangle with dimensions $(mn) \times n^2$, which gives rise to $\lfloor (mn)/n^2 \rfloor = \lfloor m/n \rfloor$ such rectangles. It also leaves two "surplus" rectangles to be dissected: one, denoted by A, with dimensions $(mn) \times (mn - \lfloor m/n \rfloor n^2)$ (this is part of the $m^2 \times n^2$ rectangle) and one, denoted by B, with dimensions $(m^2 - \lfloor m/n \rfloor mn) \times n^2$ (this is part of the $mn \times mn$ square).

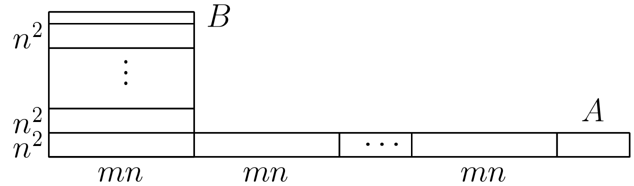

Figure 1: Step 1 in the dissection of an $m^2 \times n^2$ rectangle into an $mn \times mn$ square.

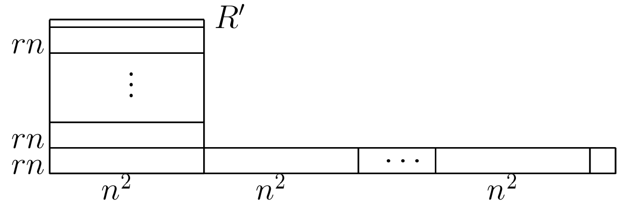

Figure 2: Step 2 of the dissection. We rotate the rectangle B and dissect. The remaining rectangle R' has dimensions $n^2 \times r^2$.

In the second step we rotate the rectangle B 90 degrees counterclockwise, The resulting rectangles have dimensions $mn \times rn$ and $n^2 \times rm$, where $r = m - \lfloor m/n \rfloor n$, We now perform the following dissection. (see Figure 2).

We dissect A into $\lfloor mn/n^2 \rfloor = \lfloor m/n \rfloor$ rectangles each of dimension $n^2 \times rn$. The remaining rectangle in A is in fact an $rn \times rn$ square. These pieces are placed in B one on top of the other. It is easy to see that the remaining rectangle has dimensions $n^2 \times r^2$.

If R' is the rectangle with dimensions $n^2 \times r^2$ we see that the original dissection problem of converting the rectangle R into a square has been transformed into the problem of converting the rectangle R' into a square at an extra cost of $2\lfloor m/n \rfloor$ rectangles. This completes the proof of Lemma 2.1. ■

Lemma 2.1 gives an algorithm for computing a dissection of the $m^2 \times n^2$ rectangle into an $mn \times mn$ square. Consider the sequence of integers generated by the Euclidean algorithm: $r_0 = m, r_1 = n$ and

$$\begin{array}{llll} r_0 & = q_0 r_1 + r_2 & & 0 \le r_2 < r_1 \\ r_1 & = q_1 r_2 + r_3 & & 0 \le r_3 < r_2 \\ \vdots & & & \vdots \\ r_i & = q_i r_{i+1} + r_{i+2} & & 0 \le r_{i+2} < r_{i+1} \\ \vdots & \vdots & & \\ r_k & = q_k r_{k+1} & & r_{k+2} = 0, \end{array}$$

where $r_{k+1} = \gcd(m, n)$ and $k \in O(\log m)$. If we iterate Lemma 2.1 k times then we obtain a dissection consisting of

$$p(m,n) \le 2 \sum_{i=0}^{k} \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor + O(\log m)$$

rectangular pieces. To sum up we have proved the following theorem.

Theorem 2.1 *An* $\frac{m}{n} \times \frac{n}{m}$ *rectangle can be dissected into a unit square using only rectilinear glass cuts. Moreover the number of pieces does not exceed*

$$2 \sum_{i=0}^{k} \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor + O(\log m),$$

where $r_0 = m > r_1 = n > \cdots > r_{k+1} = \gcd(m, n)$ *is the sequence of integers produced by the computation of* $\gcd(m, n)$ *using the Euclidean algorithm.* ∎

Note that the running time of the algorithm is terms of the number of iterations is $O(\log m)$. In general, the number of pieces obtained by the algorithm never exceeds $m + O(\log m)$. The worst-case number of pieces is obtained when the $m \times 1/m$ rectangle is dissected to form a unit square: the number of pieces required is exactly m.

3 Triangulations

In this section we consider the following problem concerning triangulations and dissections of regular convex n-gons.

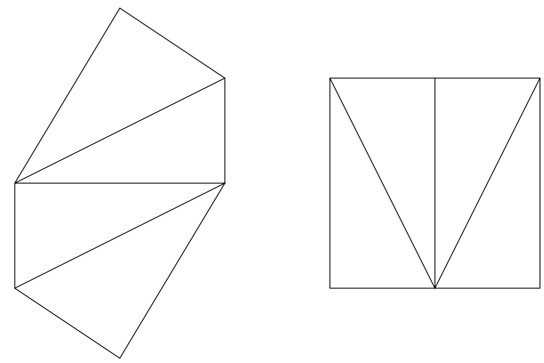

Figure 3: A triangulation of a convex polygon which is also a dissection of the square P_4.

Problem 3.1 *Can a triangulation of a regular convex* m*-gon be a dissection of another regular convex* n*-gon of the same area, for* $m \neq n$*?*

It is of course possible that an arbitrary convex polygon (i.e., not necessarily regular) has a triangulation which forms a dissection of another regular convex polygon. An example of this is depicted in Figure 3.

For each positive integer n, let P_n denote the regular convex n-gon of area equal to 1. We will prove the following result.

Lemma 3.1 *Let* $m \neq n$ *be positive integers. If a triangulation of* P_m *is a dissection of* P_n*, then*

1. n *divides* $2m$*, and*
2. $\frac{\phi(m)}{8} \le \frac{m}{n}$*,*

where ϕ *is Euler's totient function.*

Before proving the lemma we poimt out an application. Recall the well-known result of Hardy and Wright [4][Theorem 328] that

$$\liminf_{m \to \infty} \frac{\phi(m) \ln \ln m}{m} = e^{-\gamma},$$

where $\gamma = \lim_{n\to\infty}(\sum_{i=1}^{n} - \ln n)$ is Euler's constant. From this observation and our main Lemma 3.1 it follows that

Theorem 3.1 *If either $n \not| \; 2m$ or $\ln\ln m \in \Omega(n)$ then no triangulation of P_m can be a dissection of P_n.* ■

PROOF of Lemma 3.1. First we prove part (1) of the theorem. Assume we have a triangulation of P_m. Since P_m is regular it can be inscribed in a circle. This implies that each angle of each triangle of the triangulation is an integer multiple of $\frac{\pi}{m}$. Since the triangulation forms a dissection of P_n each vertex angle of the regular n-gon P_n must be a sum of angles of the triangulation. In particular, this implies that

$$\pi - \frac{2\pi}{n} = \sum_{k\in I} k\frac{\pi}{m} = \left(\sum_{k\in I} k\right)\frac{\pi}{m}$$

where I is some set of integers. It follows that $\pi - \frac{2\pi}{n} = l\frac{\pi}{m}$, for some integer $l = \sum_{k\in I} k$. Divide through by π and simplify to obtain the equation $(m-l)n = 2m$. This concludes the proof of part (1).

Now we focus on part (2) of the theorem. We compute the lengths of the diagonals of a regular m-gon of area 1. Elementary calculations show that the diagonal corresponding to an angle of size $k\pi/m$ has length exactly

$$s_k^{(m)} = 2\sin(k\pi/m)\sqrt{\frac{2}{m\sin(2\pi/m)}}.$$

By assumption the triangulation is also a dissection of P_n. This implies that the side of the regular n-gon is a sum of diagonals of P_m. Hence there exist positive integers $l_1, l_2, \ldots, l_r$ and $k_1 < k_2 < \cdots < k_r$ such that

$$s_n^{(1)} = \sum_{j=1}^{r} l_j s_m^{k_j}.$$

From this we derive the equation

$$\begin{aligned}&\textstyle\sum_{j=1}^r l_j \sin(k_j\pi/m)\\ &= \sin(\pi/n)\sqrt{\tfrac{n\sin(2\pi/n)}{m\sin(2\pi/m)}}.\end{aligned} \qquad (1)$$

Let $\omega = e^{-i\pi/m}$ denote the m-th root of unity. For any integer s we have the identity

$$\sin(s\pi/m) = \frac{e^{is\pi/m} - e^{-is\pi/m}}{2} = \frac{\omega^s - \omega^{-s}}{2}.$$

Substituting this in equation (1) and squaring both sides we obtain the equation

$$\begin{aligned}&\tfrac{m}{n}\left(\tfrac{1}{2}\textstyle\sum_{j=1}^r l_j(\omega^{k_j} - \omega^{-k_j})\right)^2\\ &= \left(\tfrac{\omega^{m/n}-\omega^{-m/n}}{2}\right)^2 \tfrac{\omega^{2m/n}-\omega^{-2m/n}}{\omega-\omega^{-1}}\end{aligned} \qquad (2)$$

Equation (2) is equivalent to a polynomial with integer coefficients of degree at most $8m/n$ which is satisfied by ω.

However, it is well-known from Galois theory that the degree of the extension field $Q(\omega)$ over Q satisfies $|Q(\omega) : Q| = \phi(m)$, where $\phi(m)$ denotes Euler's totient function. This implies that ω cannot be a root of a polynomial equation with integer coefficients of degree less than $\phi(m)$. Hence we have that $8m/n \geq \phi(m)$ which completes the proof of the second part of the lemma. ■

Lemma 3.1 shows that the answer to Problem 3.1 is negative when either n does not divide $2m$ or $\frac{n}{8} < \frac{m}{\phi(m)}$. We now consider separately the case when $n|2m$ and $\frac{n}{8} \leq \frac{m}{\phi(m)}$.

Theorem 3.2 *For n sufficiently large, no triangulation of P_{2n} can be a dissection of P_n.*

PROOF As in the proof of Theorem 3.1 the diagonals of P_{2n} have lengths given by formula

$$s_k^{(2n)} = \frac{2\sin(k\pi/2n)}{\sqrt{n\sin(\pi/n)}}$$

and the side of P_n by the formula

$$s_1^{(n)} = \frac{2\sin(\pi/n)\sqrt{2}}{\sqrt{mn\sin(2\pi/n)}}$$

It follows that

$$\frac{s_k^{(2n)}}{s_1^{(n)}} = \frac{\sin(k\pi/2n)}{\sin(\pi/n)}\sqrt{\cos(\pi/n)}, \qquad (3)$$

which converges to $k/2$ as n goes to infinity.

Assume on the contrary that a triangulation of P_{2n} is a dissection of P_n. Then a side of P_n must be the sum of diagonals of P_{2n}. We will show that this is impossible.

None of these diagonals can be equal to $s_k^{(2n)}$, for $k \geq 3$, because asymptotically in n, $\frac{s_k^{(2n)}}{s_1^{(n)}}$ is bigger than 1 whenever $k > 2$. Next we consider the cases $k \leq 2$.

Case 2. $k = 2$ then

$$\frac{s_2^{(2n)}}{s_1^{(n)}} = \sqrt{\cos(\pi/n)} < 1$$

and also converges to 1 as n goes to ∞.

Case 1. $k = 1$ then

$$\begin{aligned} \frac{s_1^{(2n)}}{s_1^{(n)}} &= \frac{\sin(\pi/2n)}{\sin(\pi/n)}\sqrt{\cos(\pi/n)} \\ &= \frac{\sqrt{\cos(\pi/n)}}{2\cos(\pi/2n)}\frac{\sqrt{2-1/\cos^2(\pi/2n)}}{2} \\ &> \frac{1}{2} \end{aligned}$$

and also converges to $1/2$ as n goes to ∞.

It follows tha $s_1^{(n)}$ cannot be the sum either of two diagonals $s_1^{(2n)}$ or of two diagonals one of the form $s_1^{(2n)}$ and one of the form $s_2^{(2n)}$. This and the previous observations prove the theorem. ■

4 Open Problems

There are several interesting combinatorial problems on dissections. One open problem is related to Problem 2.1: find a rectilinear dissection of an orthogonal polygon to a square of the same area using the optimal (or even asymptotically optimal) number of pieces.

It is not known whether or not the technique of Dehn invariants is applicable to Problem 3.1. There are several avenues to explore. E.g., P_4 can always be dissected (with not necessarily rectilinear glass cuts) to P_m and the number of pieces is asymptotically equal to $\frac{m}{2} + o(m)$ (see [6]). What is the minimal number of Steiner points (i.e. non-polygonal vertices) used?

Another interesting related question is the following decision problem: Given as input a triangulation of a simple polygon and a positive integer n, is this triangulation a dissection of a regular convex n-gon of the same area.

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