Some results on Geometric Independency Trees

Atsushi Kaneko
Kogakuin University
1-24-2 Nishi-Shinjuku, Shinjuku-ku,
Tokyo 163-8677 Japan
E-mail:*kaneko@ee.kogakuin.ac.jp*

Yoshiaki Oda
Keio University
3-14-1 Hiyoshi, Kohoku-ku,
Yokohama 223-8522 Japan
E-mail:*oda@comb.math.keio.ac.jp*

Kiyoshi Yoshimoto
Nihon University
1-8 Kanda-Surugadai, Chiyoda-ku,
Tokyo 101-8308 Japan
E-mail:*yosimoto@math.cst.nihon-u.ac.jp*

Abstract

A plane spanning tree is a tree drawn in the plane so that its edges are closed straight line segments and any two edges do not intersect internally, and no three of its vertices are collinear. In this paper, we present several results on a plane spanning tree such that the graph which is obtained from the tree by adding a line segment between any two end-vertices of the tree is self-intersecting.

1 Introduction

A geometric graph is a graph drawn in the plane by straight-line segments. Let U be a set of n points in the plane. If no three points are collinear, then we say that U is *in general position*. In this paper, we suppose that all set of points in the plane is in general position. We denote by $K(U)$ the complete geometric graph of which a vertex set is U and whose edges are straight-line segments between any two points (vertices) in U. For two points a and b in the plane, we denote by $\overline{ab}$ the closed straight line segment joining a to b. In this paper, an edge e joining a to b is the same as the straight-line segment $\overline{ab}$, and the term *line* e means the line which contains an edge e.

A non-self-intersecting spanning tree of $K(U)$, (i.e., a spanning tree T of $K(U)$ such that no two edges in $E(T)$ intersect except their common end-vertex) is said to be *a plane spanning tree (on U)*. Especially, we call a plane spanning tree with two end-vertices a *plane Hamilton path*.

Plane spanning trees have been studied by various authors. For example, Y. Ikebe et al.[4] showed that any rooted tree with n vertices can be embedded as a plane spanning tree on U, with the root being mapped onto an arbitrary specified point of U. In [5], the following theorem was presented.

Theorem 1.1 (Károlyi, Pach and Tóth) *Let U be a set of points in the plane in general position and G be a geometric graph with vertex set U. If G does not have a plane spanning tree, then the complementary geometric graph of G contains a plane spanning tree.*

E. Rivera-Campo[2] showed that a geometric graph G contains a plane spanning tree if the subgraph of G induced by any vertex subset with five vertices of G has a plane spanning tree (on the vertex subset). In this paper, we shall introduce a new kind of plane spanning tree, called a *geometric independency tree*.

Let G be a connected graph. If the end-vertices of a spanning tree of G are pairwise nonadjacent in G, then the spanning tree is called an *independency tree*.

It is a plain fact that a graph which does not have an independency tree is Hamiltonian. Furthermore, in [1] T. Böhme et al. characterized the class of graphs that contains an independency tree.

Theorem 1.2 (Böhme et al.) *A connected graph does not have an independency tree if and only if the graph is isomorphic to a cycle, a complete graph or a balanced complete bipartite graph.*

We consider a geometric version of an independency tree as follows. Let a and b be two vertices of a geometric graph G. The two vertices a and b *see each other* or *a sees b* if there does not exist an edge e in $E(G)$ such that e and the edge $\overline{ab}$ intersect internally. (Such an edge e is called a *shield between a and b*.) A *geometric independency tree (on U)* is a plane spanning tree T (on U) such that no two end-vertices of T see each other.

In Section 2, we shall determine the configurations of a set U of n points in which there does not exist a geometric independency tree on U with two and three endvertices. In Section 3, we shall show that any geometric independency tree on U does not have more than $\lfloor \frac{n}{2} \rfloor$ end-vertices. Moreover, we shall prove that given a set U of n points in the plain, there exists a geometric independency tree T on U such that T has at least $\frac{n}{6}$ end-vertices.

All notation and terminology not explained here are given in [3].

2 Characterizations of geometric independency trees with two and three end-vertices

Let conv(U) be the convex hull of a set U of points in the plane, which is the smallest convex set containing U. Denote by ∂U the set of points of U lying on the boundary of conv(U). A point of U which is not on the boundary of conv(U) is called an interior point of U. Denote by $I(U)$ the set of interior points of U. First we show the following fundamental lemma.

Lemma 2.1 *For any two points a and b in U, there exists a plane Hamilton path H with end-vertices a and b.*

We remark that any edge in such a plane Hamilton path is in conv(U). A geometric independency tree with two end-vertices is especially called an *geometric independency Hamilton path.* By using Lemma 2.1 we prove the following proposition.

Proposition 2.2 *For any two points a and b in U, there exists a geometric independency Hamilton path with end-vertices a and b if and only if a and b do not see each other on $K(U)$.*

In Theorem 3.1, we will prove that any geometric independency tree in $K(U)$ has at most $\lfloor \frac{n}{2} \rfloor$ end-vertices. By this fact, if there exists a geometric independency tree in $K(U)$, then n is at least 4. On the other hand, since the complete graph K_m with $m \geq 5$ is not planar, by the above theorem there exists a geometric independency Hamilton path for any set U with $n \geq 5$. It is obvious that if $n = 4$ and $K(U)$ is non-self-intersecting, i.e., $|\partial U| = 3$, then $K(U)$ has no geometric independency tree. Hence, we obtain the following corollary.

Corollary 2.3 *For a set U of n points in the plane with $n \geq 4$, there exists a geometric independency tree if and only if U does not satisfy the condition that $n = 4$ and $|\partial U| = 3$.*

Next, we present a stronger theorem than the above corollary, that is, there exists a geometric independency Hamilton path containing a prescribed edge with the same exception.

Theorem 2.4 *For a set U of n points in the plane with $n \geq 4$ and any edge e in $K(U)$ there exists a* *geometric independency Hamilton path which passes through e if and only if U does not satisfy the condition that $n = 4$ and $|\partial U| = 3$.*

We also show that there exists a geometric independency Hamilton path with a prescribed end-vertex in most cases.

Theorem 2.5 *For a set U of n points in the plane and any point a in U there exists a geometric independency Hamilton path H with an end-vertex a except that $n = 4$ and $|\partial U| = 3$ and that $|\partial U| = 3$, a is in ∂U and $I(U - a) = \phi$.*

We also characterized the set of points which does not have a geometric independency tree with three end-vertices.

Theorem 2.6 *For a set U of n points in the plane with $n \geq 6$ there exists a geometric independency tree with three end-vertices except that $n = 6$ and $|\partial U| = 3$ and that $n = 6$, $|\partial U| = 4$ and $|\partial(U - \{x, z\})| = |\partial(U - \{y, w\})| = 4$ where the boundary of $\mathrm{conv}(U)$ is a rectangle $xyzw$.*

3 Bounds of the number of end-vertices

In this section, we consider upper and lower bounds of the number of end-vertices in geometric independency trees. First we mention an upper bound of them.

Theorem 3.1 *Let T be a geometric independency tree of $K(U)$. Then, the number of end-vertices in T is at most $\lfloor \frac{n}{2} \rfloor$.*

In addition, this bound is tight. Let U be a set of n points in the plane such that $U = \partial U$. Set $U = \{v_1, v_2, \cdots, v_n\}$ and assume that the ordering is clockwise about the boundary of $\mathrm{conv}(U)$. Let T be the spanning tree on $K(U)$ whose edges are $\overline{v_{2i-1}v_{2i}}$ and $\overline{v_{2i-1}v_{2i+1}}$ for any integer i with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, and $\overline{v_{n-1}v_n}$ if n is even, and $\overline{v_{n-2}v_{n-1}}$ and $\overline{v_{n-1}v_n}$ if n is odd. This tree T is a geometric independency tree whose end-vertices are v_{2i} with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ and v_n.

Next, we consider a lower bound of the number of end-vertices. In section 2, we mentioned that for a set U such that $n = 4$ and $|\partial U| = 3$, $K(U)$ does not have any geometric independency tree.

Theorem 3.2 *Let U be a set of n points in the plain with $n \geq 5$. Then there exists a geometric independency tree T on U such that T has at least $\frac{n}{6}$ end-vertices.*

4 Conjecture

We give a following conjecture.

Conjecture 4.1 *Let U be a set of points and X be the set of geometric independency trees of $K(U)$, and we define $T_{\max}$ and $T_{\min}$ as follows.*

$$T_{\max} = \max_{T \in X} \#\{\text{end-vertices in } T\}$$

$$T_{\min} = \min_{T \in X} \#\{\text{end-vertices in } T\}$$

Then, $K(U)$ has a geometric independency tree with k end-vertices for any integer k with $T_{\min} \leq k \leq T_{\max}$.

References

[1] T. Böhme, H.J. Broersma, F. Göbel, A.V. Kostochka, M. Stiebitz, *Spanning trees with pairwise nonadjacent end-vertices*, Discrete Math. 170 (1997) 219-222.

[2] E. Rivera-Campo, *A note on the existance of plane spanning trees of geometric graphs*, Japan Conference on Discrete and Computational Geometry 1998.

[3] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, Inc. 1994.

[4] Y. Ikebe, M. Perles, A. Tamura and S. Tokunaga, *The rooted tree embedding problem into points in the plane,* Discrete Comput. Geom. 11 (1994), 51-63.

[5] G. Károlyi and J. Pach and G. Tóth, *Ramsey-type results for geometric graphs I*, Discrete Comput. Geom. 18 (1997) 247-255.